Description of the Tax Policy Advisers' Model*

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March 15, 2005

1 General Description

The Tax Policy Advisers Model is a large-scale overlapping generations dynamic computable general equilibrium model that can be used to analyze the short and long run economic and distributional effects of tax reform in the United States, including the transitional effects of reform on asset prices and housing values and the associated reform-induced intergenerational redistributive effects.

Consumers in the model have perfect foresight and maximize utility over a 55-year adult life, including 45 working years and 10 years of retirement. Individual lifetime utility is a discounted aggregation of utility in each of the 55 periods, which is in turn a function of leisure and an aggregate consumption good that is a composite of non-housing consumption goods, owner-occupied housing and rental housing. Bequests and inheritances are introduced into the model by assuming a fixed target bequest, and the model includes a simple characterization of Social Security.

Firms are value or profit maximizers and firm managers are also characterized by perfect foresight. Following the "q theory" of investment behavior, firm managers calculate explicitly the time path of investment in response to a change in the tax structure, taking into account convex costs of adjusting investment from its steady state level. Firm behavior is modeled for each of the three production sectors – non-housing consumption goods, owner-occupied housing and rental housing – with owner-occupiers treated as "firms" who produce housing and rent it to themselves, taking into account the tax advantages of home ownership. The debt-capital ratio is assumed to be fixed in each industry.

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The government must finance an exogenously specified time path of public services, which are assumed to be separable from the individual lifetime utility function. The government's tax instruments include a corporate income tax and an income tax with a progressive wage income tax structure and constant rate capital income taxes in the initial equilibrium. The model can analyze the short and long run effects of both reforms of the existing income tax and various kinds of consumption tax reforms, including Flat Taxes, the Bradford X-Tax, cash flow expenditure taxes, and retail sales taxes and VATs. The government must either satisfy an annual budget constraint or can issue debt for a finite period and then satisfy the annual budget constraint from that point forward.

All markets are assumed to be perfectly competitive. Market equilibrium in the model requires that total consumer demand, obtained by aggregating the demands of each of the 55 generations alive at any single point in time, must equal aggregate supply in each of the three production sectors – the non-housing consumption goods, owner-occupied housing and rental housing. In addition, factor demands must equal factor supplies in the labor and capital markets, the total amounts of debt and equity held as individual wealth must equal firm stocks of debt and equity, the government's budget must be balanced for a specified set of tax instruments, and both individual and firm expectations regarding the time paths of future prices must be satisfied.

The model calculates all asset values in all three markets explicitly for each period after the enactment of a reform, taking into account all changes in the tax treatment of existing capital assets . The model is thus especially well suited to analyzing the transitional effects of reform, including asset price effects in all three sectors, and the associated reform-induced redistributions across all generations alive at the time of reform and for future generations.

The following discussion provides the details of the economic model underlying the Tax Policy Advisers Model.

2 Timing Conventions

The model is constructed in discrete time and adopts the following conventions. All flow variables, including consumption, labor supply, saving, and investment occur at the end of the period. Individual utility levels, which are based on current and future levels of consumption and leisure, are calculated at the end of the period, and the issuance of debt and new shares to finance new investment occurs at the end of the period. The prices related to all flow variables are also determined at the end of the period. All stock variables, including asset values, capital stocks, and the stock of debt are measured at the beginning of the period. That is, stock values at the beginning of the period reflect flow values at the end of the previous period.

Present values of consumption and income for individuals are calculated at the end of period zero. Thus, period zero consumption is not discounted, and period one consumption is discounted for one full year. Since present values are taken at

the end of the period, asset values measured at the beginning of the period must be adjusted by interest for one period to be comparable with consumption, wages, and other prices, which are measured at the end of the period.

Since individual asset values are calculated at the beginning of the period, firm values must also be calculated at the beginning of the period; that is, present values for production are measured at the beginning of the period. All firm cash flow variables, including investment, sales, labor purchases, etc., occur at the end of the period. Thus, in calculating the value of the firm, the first period term is discounted, since it is calculated at the beginning of the period, evaluating cash flows that occur at the end of the period.

Births and deaths occur at the beginning of the period, as do inheritances and bequests. Thus, an individual is of age zero all of the birth period and age one all of the following period. Age a is defined at the beginning of a particular time period t and thus changes with time. For example, an individual of age $a = 10$ at the beginning of period $t = 0$ is age $a = 30$ at the beginning of period $t = 20$.

Reform is generally assumed to occur at the beginning of period $t = 1$, and to be unexpected. Reform thus affects only consumption C, saving S, and investment I decisions made at the end of period $t = 1$ and thereafter, not in period $t = 0$. This implies that period zero C_0, S_0, I_0, and capital stock K_0 and asset value A_0 as well as period $t = 1$ capital stock K_1 are fixed in the initial income tax equilibrium. The first post-reform values are thus flows C_1, S_1 and I_1, and capital stock K_2. However, asset value A_1, which is calculated at the same time reform is announced at the beginning of period $t = 1$, instantaneously reflects the anticipated asset value effects of the reform.

The q variable, which reflects the marginal value of the firm relative to replacement cost, is measured at the beginning of the period (as are firm values and capital stocks). Thus, q_0 is fixed as the value of an increase in the capital stock in period zero under the income tax regime, and q_1 is the first new value of q, measured at the beginning of period one at the same time when reform occurs. However, the first value of q relevant to new investment decisions is q_2, since end-of-period-one investment I_1 is a function of beginning of period two q_2 and not q_1.

3 Individual Behavior

Individual behavior is modeled using an overlapping generations framework that consists of 55 cohorts, denoted by ages that range from $a = 0$ to $a = 54$, as the individual economic life span is known (with certainty) to be 55 years. Each generation is represented by a single individual, who works for the first 45 of the 55-year lifespan and is retired for the last 10 years. Individual tastes are identical so that differences in behavior across generations are due solely to differences in lifetime budget constraints.

An individual of age a at the beginning of period s has assets $A_s(a)$ which have been accumulated from the time of "economic birth" and are eventually used to fund

(1) consumption over the individual's lifetime, including that during the retirement period, and (2) the bequest. Assets include inheritances, which are assumed to be received at economic age $a = 25$ (there are no separate gifts in the model). Bequests are given at the time of death, which implies that at the time of death (age $a = 55$) total assets are just sufficient to fund the bequest BQ_s which becomes the inheritance of the recipient INH_s. Following King and Fullerton (1993), the analysis utilizes the relatively simple "target model" of bequests, such that $INH_s = BQ_s = BQ_t(1+n)^{s-t}(1+g)^{s-t}$, where n denotes the population growth rate and BQ_t is the size of the bequest in some reference period t, typically the initial equilibrium ($t = 0$). Note that this implies bequests are completely insensitive to changes in economic conditions, including changes in income and rates of return.

The model assumes that individuals optimize over their life cycles and have perfect foresight. In any period s, an individual of age a chooses total consumption $\bar{C}_s(a)$ and leisure $E_s(a)$ for $s = t, ..., t + 54 - a$ to maximize rest-of-life utility LU_s,

$$LU_s(a) = \frac{1}{(1 - \frac{1}{\sigma 1})} \sum_{s=t}^{t+54-a} \frac{U_s(a)^{(1-\frac{1}{\sigma 1})}}{(1+\rho)^{s-t}}, \tag{1}$$

where $\sigma 1$ is the intertemporal elasticity of substitution, ρ is the pure rate of time preference, and $U_s(a)$ is utility in period s. Intraperiod utility is modelled as a CES function of a composite consumption good $\bar{C}_s(a)$ and leisure $E_s(a)$,

$$U_s(a) = [\alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s(a)^{\frac{\sigma 2-1}{\sigma 2}} + \alpha_E^{\frac{1}{\sigma 2}} E_s(a)^{\frac{\sigma 2-1}{\sigma 2}}]^{\frac{\sigma 2}{\sigma 2-1}}, \tag{2}$$

where $\sigma 2$ is the intratemporal elasticity of substitution between the composite consumption good and leisure in any period, α_E and α_C measure the relative intensities of household preferences for leisure and consumption (with $\alpha_E + \alpha_C = 1$), and leisure is defined as $E_s(a) = H_T - L_s(a)$, where H_T is the total number of hours available in a period for either labor supply $L_s(a)$ or leisure.

The composite consumption good is modelled as a CES function of non-housing goods and housing services,

$$\bar{C}_s(a) = [\alpha_G^{\frac{1}{\sigma 3}} C_s^{\frac{\sigma 3-1}{\sigma 3}} + \alpha_H^{\frac{1}{\sigma 3}} H_s^{\frac{\sigma 3-1}{\sigma 3}}]^{\frac{\sigma 3}{\sigma 3-1}}, \tag{3}$$

where C_s is consumption of all non-housing goods in period s, H_s is consumption of housing services in period s, α_G and α_H measure the relative intensities of household preferences for non-housing goods and housing goods (with $\alpha_G + \alpha_H = 1$), and $\sigma 3$ is the elasticity of substitution between housing and non-housing goods.[1]

The non-housing consumption good is modelled as a CES function of two goods – those generally subject to state retail sales taxes and those that are typically tax

[1] The age subscript (a) has been dropped in order to simplify the notation.

exempt – with the option of including minimum required purchases of both taxed and non-taxed goods,

$$C_s(a) = [\alpha_1^{\frac{1}{\sigma 4}}(C_{1s} - b_{1s})^{\frac{\sigma 4-1}{\sigma 4}} + \alpha_2^{\frac{1}{\sigma 4}}(C_{2s} - b_{2s})^{\frac{\sigma 4-1}{\sigma 4}}]^{\frac{\sigma 4}{\sigma 4-1}}, \tag{4}$$

where C_{is} is the consumption of good i in period s, b_{is} is the minimum required purchase of good i in period s, α_1 and α_2 measure the relative intensities of household preferences for taxed and untaxed goods (with $\alpha_1 + \alpha_2 = 1$), and $\sigma 4$ is the elasticity of substitution between "above-minimum" quantities of the taxed and non-taxed goods.

Housing services, H_s, are also modelled as a composite good specified by a CES function of the quantities of rental housing services and owner-occupied housing services in excess of the minimum required purchases,

$$H_s(a) = [\alpha_O^{\frac{1}{\sigma 5}}(OH_s - b_{Os})^{\frac{\sigma 5-1}{\sigma 5}} + \alpha_R^{\frac{1}{\sigma 5}}(RH_s - b_{Rs})^{\frac{\sigma 5-1}{\sigma 5}}]^{\frac{\sigma 5}{\sigma 5-1}}, \tag{5}$$

where $OH_s(a)$ is the quantity of owner-occupied housing services consumed by an individual of age a in period s, $RH_s(a)$ is the quantity of rental housing service consumed by an individual of age a in period s, b_{Os} and b_{Rs} represent the minimum purchase requirements of owner-occupied and rental housing, $\sigma 5$ is the elasticity of substitution between the "above-minimum" quantities of owner-occupied housing and rental housing services, and α_O and α_R measure the relative intensities of household preferences for owner-occupied and rental housing (with $\alpha_O + \alpha_R = 1$).

The individual maximizes lifetime utility $LU_s(a)$ subject to a lifetime budget constraint (recall that present values of individual consumption and income flows are taken at the end of period s) which requires that

$$\begin{aligned} 0 \;=\; & TW_s(a) - \sum_{s=t}^{t+54-a} \frac{p_s(1+\tau_{cs})(C_{1s} - b_{1s}) + p_s(C_{2s} - b_{2s})}{\prod_{u=t+1}^{s}[1 + i_u(1-\tau_{iu})]} \\ & - \sum_{s=t}^{t+54-a} \frac{p_{Os}(OH_s - b_{Os}) + p_{Rs}(RH_s - b_{Rs})}{\prod_{u=t+1}^{s}[1 + i_u(1-\tau_{iu})]} \end{aligned} \tag{6}$$

where $TW_s(a)$ is total wealth available for consumption expenditures at the end of period s for an individual age a in period s, p_s is the producer price of the consumption good — which is chosen to be the numeraire so that $p_s = 1$, τ_c is the average state retail sales tax rate applied to taxed goods, p_{Os} is the unit price of owner-occupied housing services, and p_{Rs} is the unit price of rental housing services. Note that individuals discount future cash flows at the after-tax discount rate r_s

$$r_s = i_s(1 - \tau_{is}),$$

where τ_{is} is the tax rate on interest income under the income tax in period s.

Total wealth $TW_s(a)$ available for discretionary consumption expenditures (those in excess of any minimum required consumption expenditures) for an individual of age a at the end of period s is given by

$$\begin{aligned} TW_s(a) &= A_s(a)(1+r_s) - \frac{BQ_{t+55-a}}{\prod_{u=t+1}^{t+54-a}(1+r_u)} \\ &+ \sum_{s=t}^{t+54-a} \frac{[w_s(a)(H_T - E_s)](1-\tau_{ws}-\tau_{ss}) + SSB_s}{\prod_{u=t+1}^{s}(1+r_u)} \\ &+ \sum_{s=t}^{t+54-a} \frac{TR_s + LSR_s}{\prod_{u=t+1}^{s}(1+r_u)} \\ &- \sum_{s=t}^{t+54-a} \frac{p_s(1+\tau_{cs})b_{1s} + p_s b_{2s} + p_{Os} b_{Os} + p_{Rs} b_{Rs}}{\prod_{u=t+1}^{s}(1+r_u)}, \end{aligned} \tag{7}$$

where $A_s(a)$ is the value of taxable assets (including the present value of the inheritance if not yet received and the value of individual ownership of government debt) in period s for an individual age a, BQ_{t+55-a} is the exogenous bequest, $w_s(a)$ is the wage rate earned by an individual of age a in period s, L_s is labor supplied in period s, t_{ws} is the average tax rate on wage income in period s, t_{ss} is the social security tax on wage income in period s, SSB_s represents social security benefits received in period s, TR_s is transfers received in period s (which are modeled as uniformly distributed across all 55 generations and growing at the rate of technological growth in the economy (g)), LSR_s is any lump sum rebates received at time s, and $p_{is}b_{is}$ is the cost of meeting the minimum purchase requirement for the consumption good i in period s. Following Auerbach and Kotlikoff (1987), the wage rate is defined as $w_s(a) = w_s e(a)$, where w_s is the economy-wide equilibrium wage and $h(a)$ is the individual's "human capital profile" that varies in a "hump-backed" fashion to reflect changes in productivity over the life cycle.

Dropping the notation for age to simplify notation and letting $[U^*]$, $[\bar{C}^*]$, and $[C^*]$ equal the expressions in the brackets in the definitions of U_s, $\bar{C}_s$, and C_s the first order conditions with respect to C_{1s} and C_{2s} imply

$$\frac{U_s^{\frac{-1}{\sigma 1}}[U^*]^{\frac{1}{\sigma 2-1}}\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{-1}{\sigma 2}}[\bar{C}^*]^{\frac{1}{\sigma 3-1}}\alpha_C^{\frac{1}{\sigma 3}}C^{\frac{-1}{\sigma 3}}[C^*]^{\frac{1}{\sigma 4-1}}\alpha_{C1}^{\frac{1}{\sigma 4}}(C_{1s}-b_{1s})^{\frac{-1}{\sigma 4}}}{(1+\rho)^{s-t}} = \lambda\{\frac{p_s(1+\tau_{cs})}{\prod_{u=t+1}^{s}(1+r_u)}\}$$

$$\frac{U_s^{\frac{-1}{\sigma 1}}[U^*]^{\frac{1}{\sigma 2-1}}\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{-1}{\sigma 2}}[\bar{C}^*]^{\frac{1}{\sigma 3-1}}\alpha_C^{\frac{1}{\sigma 3}}C^{\frac{-1}{\sigma 3}}[C^*]^{\frac{1}{\sigma 4-1}}\alpha_{C2}^{\frac{1}{\sigma 4}}(C_{2s}-b_{2s})^{\frac{-1}{\sigma 4}}}{(1+\rho)^{s-t}} = \lambda\{\frac{p_s}{\prod_{u=t+1}^{s}(1+r_u)}\}.$$

Taking the ratio of these two expressions yields

$$\frac{\alpha_{C1}^{\frac{1}{\sigma 4}}(C_{1s}-b_{1s})^{\frac{-1}{\sigma 4}}}{\alpha_{C2}^{\frac{1}{\sigma 4}}(C_{2s}-b_{2s})^{\frac{-1}{\sigma 4}}}=(1+\tau_{cs})$$

which can be written as

$$(C_{2s}-b_{2s})=(C_{1s}-b_{1s})\frac{\alpha_{C2}}{\alpha_{C1}}(1+\tau_{cs})^{\sigma 4}.$$

Substituting into the definition of C_s yields

$$\begin{aligned}
C_s &= [\alpha_{C1}^{\frac{1}{\sigma 4}}(C_{1s}-b_{1s})^{\frac{\sigma 4-1}{\sigma 4}}+\alpha_{C2}^{\frac{1}{\sigma 4}}(C_{2s}-b_{2s})^{\frac{\sigma 4-1}{\sigma 4}}]^{\frac{\sigma 4}{\sigma 4-1}} \\
&= [\alpha_{C1}^{\frac{1}{\sigma 4}}(C_{1s}-b_{1s})^{\frac{\sigma 4-1}{\sigma 4}}+\alpha_{C2}^{\frac{1}{\sigma 4}}\{(C_{1s}-b_{1s})\frac{\alpha_{C2}}{\alpha_{C1}}(1+\tau_{cs})^{\sigma 4}\}^{\frac{\sigma 4-1}{\sigma 4}}]^{\frac{\sigma 4}{\sigma 4-1}} \\
&= (C_{1s}-b_{1s})[\alpha_{C1}^{\frac{1}{\sigma 4}}+\alpha_{C2}^{\frac{1}{\sigma 4}}(\frac{\alpha_{C2}}{\alpha_{C1}})^{\frac{\sigma 4-1}{\sigma 4}}(1+\tau_{cs})^{\sigma 4-1}]^{\frac{\sigma 4}{\sigma 4-1}} \\
&= (C_{1s}-b_{1s})[\alpha_{C1}^{\frac{1}{\sigma 4}}+\alpha_{C2}\alpha_{C1}{}^{\frac{1-\sigma 4}{\sigma 4}}(1+\tau_{cs})^{\sigma 4-1}]^{\frac{\sigma 4}{\sigma 4-1}},
\end{aligned}$$

solving for $(C_{1s}-b_{1s})$

$$\begin{aligned}
(C_{1s}-b_{1s}) &= C_s[\alpha_{C1}^{\frac{1}{\sigma 4}}+\alpha_{C2}\alpha_{C1}{}^{\frac{1-\sigma 4}{\sigma 4}}(1+\tau_{cs})^{\sigma 4-1}]^{\frac{\sigma 4}{1-\sigma 4}}, \\
&= C_s\{\alpha_{C1}^{\frac{1-\sigma 4}{\sigma 4}}[\alpha_{C1}+\alpha_{C2}(1+\tau_{cs})^{\sigma 4-1}]\}^{\frac{\sigma 4}{1-\sigma 4}}, \\
&= C_s\alpha_{C1}[\alpha_{C1}+\alpha_{C2}(1+\tau_{cs})^{\sigma 4-1}]^{\frac{\sigma 4}{1-\sigma 4}}, \\
&= C_s\alpha_{C1}\{(1+\tau_{cs})^{\sigma 4-1}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4}+\alpha_{C2}]\}^{\frac{\sigma 4}{1-\sigma 4}}, \\
&= C_s\alpha_{C1}(1+\tau_{cs})^{-\sigma 4}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4}+\alpha_{C2}]^{\frac{\sigma 4}{1-\sigma 4}}.
\end{aligned}$$

Solving next for $(C_{2s}-b_{2s})$, recall that

$$\begin{aligned}
(C_{2s}-b_{2s}) &= (C_{1s}-b_{1s})\frac{\alpha_{C2}}{\alpha_{C1}}(1+\tau_{cs})^{\sigma 4}, \\
&= \{C_s\alpha_{C1}(1+\tau_{cs})^{-\sigma 4}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4}+\alpha_{C2}]^{\frac{\sigma 4}{1-\sigma 4}}\}\frac{\alpha_{C2}}{\alpha_{C1}}(1+\tau_{cs})^{\sigma 4}, \\
&= C_s\alpha_{C2}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4}+\alpha_{C2}]^{\frac{\sigma 4}{1-\sigma 4}}, \\
&= C_s\alpha_{C2}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4}+\alpha_{C2}]^{\frac{\sigma 4}{1-\sigma 4}}.
\end{aligned}$$

Thus, given total non-housing related consumption C_s, the allocation of consumption between taxed and untaxed goods is

$$(C_{1s}-b_{1s})=C_s\alpha_{C1}(1+\tau_{cs})^{-\sigma 4}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4}+\alpha_{C2}]^{\frac{\sigma 4}{1-\sigma 4}}, \tag{8}$$

$$(C_{2s}-b_{2s})=C_s\alpha_{C2}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4}+\alpha_{C2}]^{\frac{\sigma 4}{1-\sigma 4}}. \tag{9}$$

These two terms can be substituted into (6), the expression for total discretionary wealth, to yield

$$TW_s(a) = \sum_{s=t}^{t+54-a} \frac{F_s C_s + p_{Os}(O_s - b_{Os}) + p_{Rs}(R_s - b_{Rs})}{\prod_{u=t+1}^{s}[1 + r_u]},$$

where F_{Cs} is given by

$$F_{Cs} = p_s[\alpha_{C1}(1 + \tau_{cs})^{1-\sigma 4} + \alpha_{C2}]^{\frac{1}{1-\sigma 4}},$$

which is a function of parameters and the consumption tax rate in period s Note that when there are no untaxed goods, F_s reduces to $p_s(1 + \tau_{cs})$.

The representative agent also must choose the optimal quantity of owner-occupied and rental housing services in order to maximize lifetime utility. Dropping the notation for age and letting $[U^*]$, $[C^*]$, and $[H^*]$ equal the expressions in the brackets in the definitions of U_s, $\bar{C}_s$, and H_s the first order conditions with respect to OH_s and RH_s imply

$$\frac{U_s^{\frac{-1}{\sigma 1}}[U^*]^{\frac{1}{\sigma 2-1}}\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{-1}{\sigma 2}}[\bar{C}^*]^{\frac{1}{\sigma 3-1}}\alpha_H^{\frac{1}{\sigma 3}}H^{\frac{-1}{\sigma 3}}[H^*]^{\frac{1}{\sigma 5-1}}\alpha_O^{\frac{1}{\sigma 5}}(O_s - b_{Os})^{\frac{-1}{\sigma 5}}}{(1+\rho)^{s-t}} = \lambda\{\frac{p_{Os}}{\prod_{u=t+1}^{s}(1 + r_u)}\}$$

$$\frac{U_s^{\frac{-1}{\sigma 1}}[U^*]^{\frac{1}{\sigma 2-1}}\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{-1}{\sigma 2}}[\bar{C}^*]^{\frac{1}{\sigma 3-1}}\alpha_H^{\frac{1}{\sigma 3}}H^{\frac{-1}{\sigma 3}}[H^*]^{\frac{1}{\sigma 5-1}}\alpha_R^{\frac{1}{\sigma 5}}(R_s - b_{Rs})^{\frac{-1}{\sigma 5}}}{(1+\rho)^{s-t}} = \lambda\{\frac{p_{Rs}}{\prod_{u=t+1}^{s}(1 + r_u)}\}.$$

Taking the ratio of these two expressions yields

$$\frac{\alpha_O^{\frac{1}{\sigma 5}}(O_s - b_{Os})^{\frac{-1}{\sigma 5}}}{\alpha_R^{\frac{1}{\sigma 5}}(R_s - b_{Rs})^{\frac{-1}{\sigma 5}}} = \frac{p_{Os}}{p_{Rs}},$$

which can be written as

$$(R_s - b_{Rs}) = (O_s - b_{Os})\frac{\alpha_R}{\alpha_O}(\frac{p_{Os}}{p_{Rs}})^{\sigma 5}.$$

Substituting into the definition of H_s yields

$$\begin{aligned}
H_s &= [\alpha_O^{\frac{1}{\sigma 5}}(O_s - b_{Os})^{\frac{\sigma 5-1}{\sigma 5}} + \alpha_R^{\frac{1}{\sigma 5}}(R_s - b_{Rs})^{\frac{\sigma 5-1}{\sigma 5}}]^{\frac{\sigma 5}{\sigma 5-1}} \\
&= [\alpha_O^{\frac{1}{\sigma 5}}(O_s - b_{Os})^{\frac{\sigma 5-1}{\sigma 5}} + \alpha_R^{\frac{1}{\sigma 5}}(O_s - b_{Os})^{\frac{\sigma 5-1}{\sigma 5}}(\frac{\alpha_R}{\alpha_O})^{\frac{\sigma 5-1}{\sigma 5}}(\frac{p_{Os}}{p_{Rs}})^{\sigma 5-1}]^{\frac{\sigma 5}{\sigma 5-1}} \\
&= (O_s - b_{Os})[\alpha_O^{\frac{1}{\sigma 5}} + \alpha_R^{\frac{1}{\sigma 5}}(\frac{\alpha_R}{\alpha_O})^{\frac{\sigma 5-1}{\sigma 5}}(\frac{p_{Os}}{p_{Rs}})^{\sigma 5-1}]^{\frac{\sigma 5}{\sigma 5-1}} \\
&= (O_s - b_{Os})[\alpha_O^{\frac{1}{\sigma 5}} + \alpha_R\alpha_O^{\frac{1-\sigma 5}{\sigma 5}}(\frac{p_{Os}}{p_{Rs}})^{\sigma 5-1}]^{\frac{\sigma 5}{\sigma 5-1}}.
\end{aligned}$$

Solving for $(O_s - b_{Os})$

$$\begin{aligned}
(O_s - b_{Os}) &= H_s[\alpha_O^{\frac{1}{\sigma 5}} + \alpha_R \alpha_O{}^{\frac{1-\sigma 5}{\sigma 5}} (\frac{p_{Os}}{p_{Rs}})^{\sigma 5 - 1}]^{\frac{\sigma 5}{\sigma 5 - 1}} \\
&= H_s\{\alpha_O^{\frac{1-\sigma 5}{\sigma 5}} [\alpha_O + \alpha_R (\frac{p_{Os}}{p_{Rs}})^{\sigma 5 - 1}]\}^{\frac{\sigma 5}{1-\sigma 5}} \\
&= H_s \alpha_O [\alpha_O + \alpha_R (\frac{p_{Os}}{p_{Rs}})^{\sigma 5 - 1}]^{\frac{\sigma 5}{1-\sigma 5}} \\
&= H_s \alpha_O \{(\frac{p_{Os}}{p_{Rs}})^{\sigma 5 - 1} [\alpha_O (\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]\}^{\frac{\sigma 5}{1-\sigma 5}} \\
&= H_s \alpha_O (\frac{p_{Os}}{p_{Rs}})^{-\sigma 5} [\alpha_O (\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{\sigma 5}{1-\sigma 5}}
\end{aligned}$$

Solving next for $(R_s - b_{Rs})$, recall that

$$\begin{aligned}
(R_s - b_{Rs}) &= (O_s - b_{Os}) \frac{\alpha_R}{\alpha_O} (\frac{p_{Os}}{p_{Rs}})^{\sigma 5} \\
&= \{H_s \alpha_O (\frac{p_{Os}}{p_{Rs}})^{-\sigma 5} [\alpha_O (\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{\sigma 5}{1-\sigma 5}}\} \frac{\alpha_R}{\alpha_O} (\frac{p_{Os}}{p_{Rs}})^{\sigma 5} \\
&= H_s \alpha_R [\alpha_O (\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{\sigma 5}{1-\sigma 5}}.
\end{aligned}$$

Thus, given total housing consumption H_s, the allocation of consumption between owner-occupied and rental housing is given by

$$(O_s - b_{Os}) = H_s \alpha_O (\frac{p_{Os}}{p_{Rs}})^{-\sigma 5} [\alpha_O (\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{\sigma 5}{1-\sigma 5}} \tag{10}$$

$$(R_s - b_{Rs}) = H_s \alpha_R [\alpha_O (\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{\sigma 5}{1-\sigma 5}}. \tag{11}$$

Substituting these terms into the expression for total discretionary wealth yields

$$TW_s(a) = \sum_{s=t}^{t+54-a} \frac{F_{Cs} C_s + F_{Hs} H_s}{\prod_{u=t+1}^{s} [1 + r_u]}, \tag{12}$$

where F_{Hs},the price index for the composite housing good, is defined as

$$F_{Hs} = p_{Rs} [\alpha_O (\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{1}{1-\sigma 5}}.$$

The consumer chooses the optimal amounts of the non-housing and housing composite goods by maximizing utility with respect to C_s and H_s. The Lagrangian is given by

$$\pounds = \frac{1}{(1 - \frac{1}{\sigma 1})} \sum_{s=t}^{t+54-a} \frac{U_s{}^{(1-\frac{1}{\sigma 1})}}{(1+\rho)^{s-t}} + \lambda \{TW_s(a) - \sum_{s=t}^{t+54-a} \frac{F_{Cs} C_s + F_{Hs} H_s}{\prod_{u=t+1}^{s} [1 + r_u]}\},$$

where, repeating from above,

$$U_s = [\alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}} + \alpha_E^{\frac{1}{\sigma 2}} E_s^{\frac{\sigma 2-1}{\sigma 2}}]^{\frac{\sigma 2}{\sigma 2-1}} = [U_s^*]^{\frac{\sigma 2}{\sigma 2-1}},$$

$$\bar{C}_s = A_1[\alpha_G^{\frac{1}{\sigma 3}} C_s^{\frac{\sigma 3-1}{\sigma 3}} + \alpha_H^{\frac{1}{\sigma 3}} H_s^{\frac{\sigma 3-1}{\sigma 3}}]^{\frac{\sigma 3}{\sigma 3-1}} = [\bar{C}_s^*]^{\frac{\sigma 3}{\sigma 3-1}},$$

$$H_s = [\alpha_O^{\frac{1}{\sigma 5}} (OH_s - b_{Os})^{\frac{\sigma 5-1}{\sigma 5}} + \alpha_R^{\frac{1}{\sigma 5}} (RH_s - b_{Rs})^{\frac{\sigma 5-1}{\sigma 5}}]^{\frac{\sigma 5}{\sigma 5-1}} = [H_s^*]^{\frac{\sigma 5}{\sigma 5-1}}.$$

The first order condition with respect to C_s is

$$\frac{[U_s^*]^{[\frac{-1}{\sigma 1}\frac{\sigma 2}{\sigma 2-1}]}[U_s^*]^{[\frac{1}{\sigma 2}\frac{\sigma 2}{\sigma 2-1}]}\alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s^{\frac{-1}{\sigma 2}} [\bar{C}_s^*]^{[\frac{1}{\sigma 3-1}]}\alpha_G^{\frac{1}{\sigma 3}} C_s^{\frac{-1}{\sigma 3}}}{(1+\rho)^{s-t}} = \lambda\{\frac{F_{Cs}}{\prod_{u=t+1}^{s}[1+r_u]}\}$$

$$\frac{[U_s^*]^{[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}\alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s^{\frac{-1}{\sigma 2}} [C_s^*]^{[\frac{1}{\sigma 3-1}]}\alpha_G^{\frac{1}{\sigma 3}} C_s^{\frac{-1}{\sigma 3}}}{(1+\rho)^{s-t}} = \lambda\{\frac{F_{Cs}}{\prod_{u=t+1}^{s}[1+r_u]}\}$$

while, similarly, the first order condition with respect to H_s is

$$\frac{[U_s^*]^{[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}\alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s^{\frac{-1}{\sigma 2}} [\bar{C}_s^*]^{[\frac{1}{\sigma 3-1}]}\alpha_H^{\frac{1}{\sigma 3}} H_s^{\frac{-1}{\sigma 3}}}{(1+\rho)^{s-t}} = \lambda\{\frac{F_{Hs}}{\prod_{u=t+1}^{s}[1+r_u]}\}.$$

Taking the ratio of these two conditions yields

$$\frac{\alpha_G^{\frac{1}{\sigma 3}} C_s^{\frac{-1}{\sigma 3}}}{\alpha_H^{\frac{1}{\sigma 3}} H_s^{\frac{-1}{\sigma 3}}} = \frac{F_{Cs}}{F_{Hs}}.$$

Solving for H_s

$$H_s = (\frac{F_{Cs}}{F_{Hs}})^{\sigma 3} \frac{\alpha_H}{\alpha_G} C_s,$$

and substituting into the definition of $\bar{C}_s$ yields

$$\begin{aligned}
\bar{C}_s &= A_1[\alpha_G^{\frac{1}{\sigma 3}} C_s^{\frac{\sigma 3-1}{\sigma 3}} + \alpha_H^{\frac{1}{\sigma 3}} H_s^{\frac{\sigma 3-1}{\sigma 3}}]^{\frac{\sigma 3}{\sigma 3-1}} \\
&= A_1[\alpha_G^{\frac{1}{\sigma 3}} C_s^{\frac{\sigma 3-1}{\sigma 3}} + \alpha_H^{\frac{1}{\sigma 3}} [(\frac{F_{Cs}}{F_{Hs}})^{\sigma 3} \frac{\alpha_H}{\alpha_G} C_s]^{\frac{\sigma 3-1}{\sigma 3}}]^{\frac{\sigma 3}{\sigma 3-1}} \\
&= C_s A_1[\alpha_G^{\frac{1}{\sigma 3}} + \alpha_H \alpha_G^{\frac{1-\sigma 3}{\sigma 3}} (\frac{F_{Cs}}{F_{Hs}})^{\sigma 3-1}]^{\frac{\sigma 3}{\sigma 3-1}} \\
&= C_s A_1[\alpha_G^{\frac{1}{\sigma 3}} + \alpha_H \alpha_G^{\frac{1-\sigma 3}{\sigma 3}} (\frac{F_{Cs}}{F_{Hs}})^{\sigma 3-1}]^{\frac{\sigma 3}{\sigma 3-1}}.
\end{aligned}$$

Solving for C_s yields

$$\begin{aligned} C_s &= \frac{\bar{C}_s}{A_1}[\alpha_G^{\frac{1}{\sigma 3}} + \alpha_H \alpha_G^{\frac{\sigma 3}{1-\sigma 3}} (\frac{F_{Cs}}{F_{Hs}})^{\sigma 3 - 1}]^{\frac{\sigma 3}{1-\sigma 3}} \\ &= \frac{\bar{C}_s}{A_1}\{\alpha_G^{\frac{1-\sigma 3}{\sigma 3}} (\frac{F_{Cs}}{F_{Hs}})^{\sigma 3 - 1}[\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]\}^{\frac{\sigma 3}{1-\sigma 3}} \\ &= \frac{\bar{C}_s}{A_1}\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{-\sigma 3}[\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{\sigma 3}{1-\sigma 3}}. \end{aligned}$$

Solving next for H_s, recall that

$$\begin{aligned} H_s &= (\frac{F_{Cs}}{F_{Hs}})^{\sigma 3} \frac{\alpha_H}{\alpha_G} C_s, \\ &= (\frac{F_{Cs}}{F_{Hs}})^{\sigma 3} \frac{\alpha_H}{\alpha_G} \frac{\bar{C}_s}{A_1} \alpha_G (\frac{F_{Cs}}{F_{Hs}})^{-\sigma 3}[\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{\sigma 3}{1-\sigma 3}} \\ &= \frac{\bar{C}_s}{A_1}\alpha_H [\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{\sigma 3}{1-\sigma 3}}. \end{aligned}$$

Thus, given total composite consumption $\bar{C}_s$, the allocation of consumption between non-housing and housing goods is given by

$$C_s = \frac{\bar{C}_s}{A_1}\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{-\sigma 3}[\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{\sigma 3}{1-\sigma 3}}, \tag{13}$$

$$H_s = \frac{\bar{C}_s}{A_1}\alpha_H [\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{\sigma 3}{1-\sigma 3}}. \tag{14}$$

Substituting these terms into the expression for total discretionary wealth yields

$$TW_s(a) = \sum_{s=t}^{t+54-a} \frac{F_{Cs}C_s + F_{Hs}H_s}{\prod_{u=t+1}^{s}[1+r_u]},$$

$$TW_s(a) = \sum_{s=t}^{t+54-a} \frac{\frac{\bar{C}_s}{A_1}[F_{Cs}\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{-\sigma 3} + F_{Hs}\alpha_H][\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{\sigma 3}{\sigma 3 - 1}}}{\prod_{u=t+1}^{s}[1+r_u]},$$

$$TW_s(a) = \sum_{s=t}^{t+54-a} \frac{\frac{\bar{C}_s}{A_1}F_{Hs}[\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H][\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{\sigma 3}{\sigma 3 - 1}}}{\prod_{u=t+1}^{s}[1+r_u]},$$

$$TW_s(a) = \sum_{s=t}^{t+54-a} \frac{F_s \bar{C}_s}{\prod_{u=t+1}^{s}[1+r_u]},$$

where

$$F_s = \frac{F_{Hs}[\alpha_G (\frac{F_{Cs}}{F_{Hs}})^{1-\sigma 3} + \alpha_H]^{\frac{1}{\sigma 3 - 1}}}{A_1},$$

and, repeating

$$F_{Cs} = p_s[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 4} + \alpha_{C2}]^{\frac{1}{1-\sigma 4}},$$
$$F_{Hs} = p_{Rs}[\alpha_O(\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{1}{1-\sigma 5}}.$$

The Lagrangian for the consumer optimization problem of choosing $\bar{C}_s$ and E_s is

$$\pounds = \frac{1}{(1-\frac{1}{\sigma 1})}\sum_{s=t}^{t+54-a}\frac{U_s{}^{(1-\frac{1}{\sigma 1})}}{(1+\rho)^{s-t}} + \lambda\{TW_s - \sum_{s=t}^{t+54-a}\frac{F_s\bar{C}_s}{\prod_{u=t+1}^{s}[1+r_u]}\}.$$

The first order condition with respect to $\bar{C}_s$ is

$$\frac{[U_s^*]^{[\frac{-1}{\sigma 1}\frac{\sigma 2}{\sigma 2-1}]}[U_s^*]^{[\frac{1}{\sigma 2}\frac{\sigma 2}{\sigma 2-1}]}\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{-1}{\sigma 2}}}{(1+\rho)^{s-t}} = \lambda\{\frac{F_s}{\prod_{u=t+1}^{s}[1+r_u]}\}$$

$$\frac{[U_s^*]^{[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{-1}{\sigma 2}}}{(1+\rho)^{s-t}} = \lambda\{\frac{F_s}{\prod_{u=t+1}^{s}[1+r_u]}\}$$

while, similarly, the first order condition with respect to E_s is

$$\frac{[U_s^*]^{[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}\alpha_E^{\frac{1}{\sigma 2}}E_s^{\frac{-1}{\sigma 2}}}{(1+\rho)^{s-t}} = \lambda\{\frac{w_s(1-\tau_{ws}-\tau_{ss})}{\prod_{u=t+1}^{s}[1+r_u]}\}.$$

Taking the ratio of these two conditions yields

$$\frac{\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{-1}{\sigma 2}}}{\alpha_E^{\frac{1}{\sigma 2}}E_s^{\frac{-1}{\sigma 2}}} = \frac{F_s}{w_s(1-\tau_{ws}-\tau_{ss})}.$$

Solving for E_s yields

$$E_s = \{\frac{F_s}{w_s(1-\tau_{ws}-\tau_{ss})}\}^{\sigma 2}(\frac{\alpha_E}{\alpha_C})\bar{C}_s.$$

Substituting into $[U_s^*]$ yields

$$\begin{aligned}
[U_s^*] &= [\alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}} + \alpha_E^{\frac{1}{\sigma 2}}E_s^{\frac{\sigma 2-1}{\sigma 2}}] \\
&= \alpha_C^{\frac{1}{\sigma 2}}\bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}} + \alpha_E^{\frac{1}{\sigma 2}}[\{\frac{F_s}{w_s(1-\tau_{ws}-\tau_{ss})}\}^{\sigma 2}(\frac{\alpha_E}{\alpha_C})\bar{C}_s]^{\frac{\sigma 2-1}{\sigma 2}} \\
&= \{\alpha_C^{\frac{1}{\sigma 2}} + \alpha_E[\{\frac{F_s}{w_s(1-\tau_{ws}-\tau_{ss})}\}^{\sigma 2}(\frac{1}{\alpha_C})]^{\frac{\sigma 2-1}{\sigma 2}}\}\bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}} \\
&= \{\alpha_C^{\frac{1}{\sigma 2}} + \alpha_C^{(\frac{1}{\sigma 2}-1)}\alpha_E[\frac{F_s}{w_s(1-\tau_{ws}-\tau_{ss})}]^{(\sigma 2-1)}\}\bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}} \\
&= \{\alpha_C + \alpha_E[\frac{w_s(1-\tau_{ws}-\tau_{ss})}{F_s}]^{(1-\sigma 2)}\}\alpha_C^{(\frac{1}{\sigma 2}-1)}\bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}}
\end{aligned}$$

or

$$[U_s^*] = FU_s \alpha_C^{(\frac{1}{\sigma 2}-1)} \bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}}$$

where

$$FU_s = \alpha_C + \alpha_E [\frac{w_s(1-\tau_{ws}-\tau_{ss})}{F_s}]^{(1-\sigma 2)}.$$

Substituting back into the first order condition for $\bar{C}_s$ yields

$$\frac{[U_s^*]^{[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]} \alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s^{(\frac{-1}{\sigma 2})}}{(1+\rho)^{s-t}} = \lambda \{ \frac{F_s}{\prod_{u=t+1}^{s}[1+r_u]} \}$$

$$\frac{[FU_s \alpha_C^{(\frac{1}{\sigma 2}-1)} \bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}}] \alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s^{(\frac{-1}{\sigma 2})}}{(1+\rho)^{s-t}} = \lambda \{ \frac{F_s}{\prod_{u=t+1}^{s}[1+r_u]} \}$$

Thus, the first order conditions for the composite consumption good in any two periods $\bar{C}_s$ and $\bar{C}_{s-1}$imply that

$$\frac{[FU_s \alpha_C^{(\frac{1}{\sigma 2}-1)} \bar{C}_s^{\frac{\sigma 2-1}{\sigma 2}}] \alpha_C^{\frac{1}{\sigma 2}} \bar{C}_s^{(\frac{-1}{\sigma 2})}}{(1+\rho)^{s-t}} = \lambda \{ \frac{F_s}{\prod_{u=t+1}^{s}[1+r_u)]} \}$$

$$\frac{[FU_{s-1} \alpha_C^{(\frac{1}{\sigma 2}-1)} \bar{C}_{s-1}^{\frac{\sigma 2-1}{\sigma 2}}] \alpha_C^{\frac{1}{\sigma 2}} \bar{C}_{s-1}^{(\frac{-1}{\sigma 2})}}{(1+\rho)^{s-t-1}} = \lambda \{ \frac{F_{s-1}}{\prod_{u=t+1}^{s-1}[1+r_u]} \}$$

Taking the ratio of these two equations and multiplying by $(1+\rho)$ yields

$$\frac{\bar{C}_s{}^{(\frac{-1}{\sigma 1})}}{\bar{C}_{s-1}{}^{(\frac{-1}{\sigma 1})}} \frac{FU_s{}^{[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}}{FU_{s-1}{}^{[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}} = \frac{(1+\rho)}{(1+r_s)} \frac{F_s}{F_{s-1}}$$

$$\frac{\bar{C}_s{}^{(\frac{-1}{\sigma 1})}}{\bar{C}_{s-1}{}^{(\frac{-1}{\sigma 1})}} = \frac{(1+\rho)}{(1+r_s)} \frac{F_s}{F_{s-1}} \frac{FU_s{}^{-[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}}{FU_{s-1}{}^{-[(\frac{1}{\sigma 2}-\frac{1}{\sigma 1})\frac{\sigma 2}{\sigma 2-1}]}}$$

$$\frac{\bar{C}_s}{\bar{C}_{s-1}} = [\frac{(1+r_s)}{(1+\rho)} \frac{F_{s-1}}{F_s}]^{\sigma 1} \frac{FU_s{}^{[\frac{\sigma 1-\sigma 2}{\sigma 2-1}]}}{FU_{s-1}{}^{[\frac{\sigma 1-\sigma 2}{\sigma 2-1}]}}$$

which implies the optimal time path of $\bar{C}_s(a)$

$$\begin{aligned} \bar{C}_s(a) &= \{[\frac{(1+r_s)}{(1+\rho)} \frac{F_{s-1}}{F_s}]^{\sigma 1} \frac{FU_s{}^{[\frac{\sigma 2-\sigma 1}{1-\sigma 2}]}}{FU_{s-1}{}^{[\frac{\sigma 2-\sigma 1}{1-\sigma 2}]}}\} \bar{C}_{s-1}(a) \\ &= \{[\frac{(1+r_s)}{(1+\rho)} \frac{F_{s-1}}{F_s}]^{\sigma 1} RFU_s\} \bar{C}_{s-1}(a) \end{aligned} \tag{15}$$

where

$$RFU_s = [\frac{FU_s}{FU_{s-1}}]^{[\frac{\sigma 2-\sigma 1}{1-\sigma 2}]}.$$

Substituting (15) into (12) repeatedly and continuing this process until $s = t$ yields

$$\begin{aligned}
TW_t(a) &= \sum_{s=t}^{t+54-a} \frac{F_s\{\frac{(1+r_s)}{(1+\rho)}\frac{F_{s-1}}{F_s}RFU_s\}^{\sigma 1}\bar{C}_{s-1}(a)}{\prod_{u=t+1}^{s}[1+r_u]} \\
&= \sum_{s=t}^{t+54-a} \frac{(1+\rho)^{-\sigma 1}(F_{s-1})^{\sigma 1}(F_s)^{(1-\sigma 1)}RFU_s\bar{C}_{s-1}(a)}{[1+r_s]^{(1-\sigma 1)}\prod_{u=t+1}^{s-1}[1+r_u]} \\
&= \sum_{s=t}^{t+54-a} \frac{(1+\rho)^{-\sigma 1}(F_{s-1})^{\sigma 1}(F_s)^{(1-\sigma 1)}RFU_s\{\frac{[1+r_{s-1}]}{(1+\rho)}\frac{F_{s-2}}{F_{s-1}}RFU_{s-1}\}^{\sigma 1}\bar{C}_{s-2}(a)}{[1+r_s]^{(1-\sigma 1)}\prod_{u=t+1}^{s-1}[1+r_u]} \\
&= \sum_{s=t}^{t+54-a} \frac{(1+\rho)^{-2\sigma 1}(F_t)^{\sigma 1}(F_s)^{(1-\sigma 1)}}{\prod_{u=s-1}^{s}[1+r_u]^{(1-\sigma 1)}\prod_{u=t+1}^{s-2}[1+r_u]}(\prod_{u=s-1}^{s}RFU_u)\bar{C}_{s-2}(a)...
\end{aligned}$$

This can be written as

$$TW_t(a) = \{\sum_{s=t}^{t+54-a} \frac{(1+\rho)^{-(s-t)\sigma 1}(F_{s-2})^{\sigma 1}(F_s)^{(1-\sigma 1)}}{\prod_{u=t+1}^{s}[1+r_u]^{(1-\sigma 1)}}\}(\prod_{u=t+1}^{s}RFU_u)\bar{C}_t(a).$$

Defining

$$\Psi_t(a) = \{\sum_{s=t}^{t+54-a} \frac{(1+\rho)^{-(s-t)\sigma}(F_{s-2})^{\sigma}(F_s)^{1-\sigma}}{\prod_{u=t+1}^{s}[1+i_u(1-\tau_{iu})]^{1-\sigma}}(\prod_{u=t+1}^{s}RFU_u)\}$$

implies

$$\bar{C}_t(a) = \frac{TW_t(a)}{\Psi_t(a)}. \tag{16}$$

The consumption path for the remainder of the life for an individual age a is specified by 16, and the consumption levels of taxed non-housing goods, untaxed non-housing goods, owner-occupied housing, and rental housing are thus given by

$$\begin{aligned}
C_{1s} - b_{1s} &= \bar{C}_s\alpha_{C1}[\alpha_{C1} + \alpha_{C2}(1+\tau_{cs})^{\sigma 3-1}(\frac{p_s}{p_s})^{\sigma 3-1}]^{(\frac{\sigma 3}{1-\sigma 3})}, \\
C_{2s} - b_{2s} &= \bar{C}_s\alpha_{C2}[\alpha_{C1}(1+\tau_{cs})^{1-\sigma 3}(\frac{p_s}{p_s})^{1-\sigma 3} + \alpha_{C2}]^{(\frac{\sigma 3}{1-\sigma 3})},
\end{aligned}$$

$$\begin{aligned}
(O_s - b_{Os}) &= H_s\alpha_O[\alpha_O + \alpha_R(\frac{p_{Os}}{p_{Rs}})^{\sigma 5-1}]^{\frac{\sigma 5}{1-\sigma 5}}, \\
(R_s - b_{Rs}) &= H_s\alpha_R[\alpha_O(\frac{p_{Os}}{p_{Rs}})^{1-\sigma 5} + \alpha_R]^{\frac{\sigma 5}{1-\sigma 5}}.
\end{aligned}$$

3.1 The Tax System

The income tax is modeled as a progressive tax on labor income, coupled with flat rate taxes on capital income. The total tax burden on taxable wage income is defined as $[\psi + (\frac{\chi}{2})(w_s L_s)](w_s L_s)$, where $\chi > 0$ and $w_s L_s$is the individual's wage tax base in period s. This implies an average tax rate on labor income of

$$\tau_{wavgs} = \psi + (\frac{\chi}{2})(w_s L_s)$$

and a marginal tax rate of

$$\tau_{ws} = \psi + \chi(w_s L_s).$$

When $\chi = 0$, the income tax system is proportional.

Capital income is assumed to be taxed at flat rates τ_{ds} on dividends, τ_{is} on interest and τ_{gs} on capital gains. The tax rate on capital gains is an effective annual accrual rate, taking into account the benefits of tax deferral until gains are realized and tax exemption of gains transferred at death.

3.2 Social Security

The social security benefit, SSB_s, is received by those who reach an economic age of $a = 45$ in year s and continues until death at age $a = 54$ The benefit payment depends on average wage earnings over the 45 working years and the assumed social security replacement rate, RSS . Average wage earnings over the first 45 working years of the individuals life equal

$$AVE_s = \sum_{a=1}^{45} w_{s-45+a}\ h_{s-45+a}\ Le_{s-45+a}/45$$

where w_{s-45+a} represents the wage in year $s + 45 - a$, h_{s-45+a} is the individual's human capital profile factor in that same year, and Le_{s-45+a} represents effective labor in year $s - 45 + a$ of individuals that reached age 45 in year s (reflecting constant productivity growth at rate g). The social security benefit received is the product of the replacement rate and average earnings, or

$$SSB_s = RSS\ AVE_s.$$

Assuming that social security is self financing (in each period) implies that annual social security taxes must equal annual benefit payments:

$$\frac{\tau_{ss} \sum_{a=1}^{45} w_{s,a}\ h_{s,a}\ Le_{s,a}}{(1+n)^{(a-1)}} = \frac{SSB_{s-u}}{(1+n)^{(45+u)}}.$$

4 Firm Behavior: The Non-Housing Production Sector

The Tax Policy Advisers model has three production sectors – the composite consumption or non-housing good, owner-occupied housing and rental housing. In each sector, the analysis assumes that firm managers (owner-occupiers in the case of owner-occupied housing) act to maximize the value of the firm in a perfectly competitive environment in the absence of uncertainty. The approach utilized is based on Tobin's "q" theory of investment, as extended to include adjustment costs by Hayashi (1982). It is similar to the firm modeling approaches used by Goulder and Summers (1989) and Keuschnigg (1990).

The non-housing production sector is characterized by a Cobb-Douglas production function

$$F(K_s, Le_s) = (K_s)^{\alpha_1}(Le_s)^{1-\alpha_1}, \qquad 0 < \alpha_1 < 1, \tag{17}$$

where K_s denotes inputs of capital used for production in period s, Le_s denotes effective labor used for non-housing production in period s, and α_1 is the capital share parameter in the Cobb-Douglas production function.

The size of the population is assumed to grow at a constant rate n. Thus, the size of the population of age a in year s, $P_s(a)$, is

$$P_s(a) = P_t(a)(1+n)^{s-t}.$$

where $P_t(a)$ is the size of this population in some reference time period, typically the initial equilibrium ($t = 0$). This implies that in any period s, the total effective labor force is

$$Le_s = \sum_{b=0}^{44} P_s(a)L_s(a) \tag{18}$$

Initial conditions for Le_s and P_s are established at the beginning of period zero ($s = 0$).

Gross investment in period s, I_s, equals

$$I_s = K_{s+1} - (1-\delta)K_s \tag{19}$$

where δ is the rate of depreciation of the capital stock. The investment good and the non-housing consumption good are assumed to be identical (i.e., produced using a single production function).

Following Goulder and Summers (1989), adjustment costs per unit of investment are assumed to be

$$\Phi_s(\frac{I_s}{K_s}) = \frac{p_s(\frac{\beta}{2})(\frac{I_s}{K_s} - \mu)^2}{(\frac{I_s}{K_s})} \tag{20}$$

where β and μ are the adjustment cost parameters; higher values of β and lower values of μ imply higher adjustment costs. The value of μ is set equal to the steady state ratio of gross investment to capital; that is, $\mu = \delta + n + g + ng$.

The price of the non-housing production good, p_s, is assumed to be the numeriare. Note that for this adjustment cost function, the total derivative with respect to the ratio of investment to capital, $\frac{I_s}{K_s}$, is

$$\Phi_s'(\frac{I_s}{K_s}) = \frac{d\Phi_s}{d(\frac{I_s}{K_s})} = p_s(\frac{\beta}{2})[1 - (\frac{\mu}{\frac{I_s}{K_s}})^2], \tag{21}$$

and the partial derivatives with respect to investment and the capital stock are

$$\frac{\partial \Phi_s}{\partial I_s} = p_s(\frac{\beta}{2K_s})[1 - (\frac{\mu}{\frac{I_s}{K_s}})^2] = \frac{\Phi_s'}{K_s},$$

and

$$\frac{\partial \Phi_s}{\partial K_s} = -p_s(\frac{\beta}{2K_s})(\frac{I_s}{K_s^2})[1 - (\frac{\mu}{\frac{I_s}{K_s}})^2] = -(\frac{I_s}{K_s^2})\Phi_s'. \tag{22}$$

Finally, for future reference, note that

$$\begin{aligned} \Phi_s + (\frac{I_s}{K_s})\Phi_s' &= p_s(\frac{\beta}{2})\frac{(\frac{I_s}{K_s} - \mu)^2}{(\frac{I_s}{K_s})} + p_s(\frac{\beta}{2})(\frac{I_s}{K_s})[1 - (\frac{\mu}{\frac{I_s}{K_s}})^2] \\ &= p_s[(\frac{\beta}{2})(\frac{I_s}{K_s}) - \mu\beta + (\frac{\beta}{2})(\frac{\mu^2}{\frac{I_s}{K_s}}) + (\frac{\beta}{2})(\frac{I_s}{K_s}) - (\frac{\beta}{2})(\frac{\mu^2}{\frac{I_s}{K_s}}) \\ &= p_s\beta(\frac{I_s}{K_s}) - \mu\beta \\ &= p_s\beta[(\frac{I_s}{K_s}) - \mu]. \end{aligned} \tag{23}$$

Investment can be financed with either debt or equity. Firms are assumed to maintain a constant debt-capital ratio, b, so that

$$B_s = bK_s, \tag{24}$$

where B_s is the stock of outstanding debt at time s. New bond issues in period s, BN_s, are the difference in bonds outstanding in two consecutive periods, or $BN_s = B_{s+1} - B_s$. Combining equations (19) and (24) yields

$$BN_s = b(K_{s+1} - K_s) = b(I_s - \delta K_s) \tag{25}$$

Note that this formulation implies that existing loans are repaid at the rate of depreciation of the existing capital stock; that is, loans outstanding in period $s+1$ are $B_{s+1} = bK_{s+1} = bK_s - b\delta K_s + bI_s$, so that the existing stock of debt outstanding

equals bK_s, δbK_s is the amount of existing debt that is retired in period s, and bI_s is the amount of new debt that is accumulated in period s.

Total equity earnings (before depreciation) in period s, $EARN_s$, are defined as the value of output less labor costs and real interest payments on the total level of indebtedness B_s, or

$$EARN_s = p_s F(K_s, Le_s) - w_s Le_s - i_s B_s. \tag{26}$$

Dividends paid are assumed to equal a constant fraction (ζ) of the firm's after-tax earnings net of economic depreciation, or

$$DIV_s = \zeta(EARN_s - TE_s - p_s \delta K_s) \tag{27}$$

where TE_s denotes total corporate taxes paid in period s. Assuming that adjustment costs are fully deductible and that the corporate business tax rate is τ_{bs}, total corporate taxes are defined as

$$TE_s = \tau_{bs}[p_s F(K_s, Le_s) - w_s L_s - f_1 I_s(1+\Phi_s) - f_2 i_s B_s + f_3 bI_s - f_4 \delta bK_s - f_5 \delta^\tau K_s^\tau]. \tag{28}$$

where δ^τ is the depreciation rate for tax purposes and K_s^τ is tax basis under the income tax. This formulation accommodates a wide variety of alternative tax regimes. Under an income tax, with deductions for depreciation and interest expense, $f_2 = f_5 = 1$ and $f_1 = f_3 = f_4 = 0$. Under a "R-based" cash flow business tax which ignores loan transactions but allows expensing $f_1 = 1$ and $f_2 = f_3 = f_4 = f_5 = 0$. Under a "R+F based" cash flow business tax which allows expensing and treats loans on a cash flow basis (including the proceeds of the loan in the tax base while allowing deductions for repayment of interest and principal), $f_5 = 0$ and $f_1 = f_2 = f_3 = f_4 = 1$. Assuming no cash accumulation on the part of the firm, cash inflows in period s must equal total disbursements, or

$$EARN_s + BN_s + VN_s = DIV_s + I_s(1 + \Phi_s) + TE_s. \tag{29}$$

Following Goulder and Summers (1989), the model assumes individual level arbitrage. The after-tax nominal return on bonds is $r_s = (1 - \tau_{is})i_s$ so that

$$r_s = (1 - \tau_{is})i_s = \frac{(1 - \tau_{ds})DIV_s + (1 - \tau_{gs})(V_{s+1} - V_s - VN_s)}{V_s}; \tag{30}$$

that is, the effective tax rate on equity income at the individual level is a weighted average of the tax rate on dividends τ_d, the effective annual accrual tax rate on capital gains τ_g (taking into account the benefits of deferral and exemption of gains transferred at death and the cost of the taxation of nominal gains), V_s is the value of the firm, VN_s is new share issues, $(V_{s+1} - V_s - VN_s)$ is the capital gain on outstanding shares, and DIV_s is dividends paid. This treatment of equity finance follows the "traditional" view of the effects of dividend taxation. As noted above, dividends are

a fixed fraction of earnings after taxes and depreciation. Investments are financed from the remaining retained earnings, or with new share issues if retained earnings are insufficient to finance the desired level of investment. (If desired investment is less than retained earnings, the firm repurchases shares without paying a dividend tax.)

Rearranging and simplifying equation (30) yields

$$(1-\tau_{gs})V_{s+1} = (1-\tau_{gs})(V_s + VN_s) - (1-\tau_{ds})DIV_s + [(1-\tau_{is})i_s]V_s,$$

$$V_{s+1} = V_s(1+\theta_s) + VN_s - [\frac{1-\tau_{ds}}{1-\tau_{gs}}]DIV_s, \tag{31}$$

where $\theta_s = \frac{(1-\tau_{is})i_s}{(1-\tau_{gs})}$. Equation (31) can be used to obtain the expression for the value of the firm by repeatedly substituting for V_{s+i-1} as follows. Note that V_{s+2} can be written as

$$V_{s+2} = V_{s+1}(1+\theta_{s+1}) + VN_{s+1} - [\frac{(1-\tau_{ds+1})}{(1-\tau_{gs+1})}]DIV_{s+1},$$

so that substituting for V_{s+1} yields

$$\begin{aligned} V_{s+2} &= \{V_s(1+\theta_s) + VN_s - [\frac{(1-\tau_{ds})}{(1-\tau_{gs})}]DIV_s\}(1+\theta_{s+1}) \\ &\quad +VN_{s+1} - [\frac{(1-\tau_{ds+1})}{(1-\tau_{gs+1})}]DIV_{s+1}, \end{aligned}$$

$$\begin{aligned} V_{s+2} &= V_s(1+\theta_s)(1+\theta_{s+1}) + VN_s(1+\theta_{s+1}) + VN_{s+1} \\ &\quad -[\frac{(1-\tau_{ds})}{(1-\tau_{gs})}]DIV_s(1+\theta_{s+1}) - [\frac{(1-\tau_{ds+1})}{(1-\tau_{gs+1})}]DIV_{s+1}. \end{aligned}$$

Similarly, V_{s+3} can be written as

$$V_{s+3} = V_{s+2}(1+\theta_{s+2}) + VN_{s+2} - [\frac{(1-\tau_{ds+2})}{(1-\tau_{gs+2})}]DIV_{s+2},$$

so that substituting for V_{s+2} yields

$$\begin{aligned} V_{s+3} &= \{V_s(1+\theta_s)(1+\theta_{s+1}) + VN_s(1+\theta_{s+1}) + VN_{s+1} \\ &\quad -[\frac{(1-\tau_{ds})}{(1-\tau_{gs})}]DIV_s(1+\theta_{s+1}) - [\frac{(1-\tau_{ds+1})}{(1-\tau_{gs+1})}]DIV_{s+1}\}(1+\theta_{s+2}) \\ &\quad +VN_{s+2} - [\frac{(1-\tau_{ds+2})}{(1-\tau_{gs+2})}]DIV_{s+2}, \end{aligned}$$

$$
\begin{aligned}
V_{s+3} &= V_s(1+\theta_s)(1+\theta_{s+1})(1+\theta_{s+2}) + VN_s(1+\theta_{s+1})(1+\theta_{s+2}) \\
&\quad + VN_{s+1}(1+\theta_{s+2}) + VN_{s+2} - [\frac{(1-\tau_{ds})}{(1-\tau_{gs})}]DIV_s(1+\theta_{s+1})(1+\theta_{s+2}) \\
&\quad - [\frac{(1-\tau_{ds+1})}{(1-\tau_{gs+1})}]DIV_{s+1}(1+\theta_{s+2}) - [\frac{(1-\tau_{ds+2})}{(1-\tau_{gs+2})}]DIV_{s+2}.
\end{aligned}
$$

Continuing this process indefinitely yields

$$
V_{T+1} = V_s \prod_{v=s}^{T}(1+\theta_v) + \sum_{u=s}^{T} VN_u \prod_{j=s+1}^{T}(1+\theta_j) - \sum_{u=s}^{T}[\frac{(1-\tau_{du})}{(1-\tau_{gu})}]DIV_u \prod_{j=s+1}^{T}(1+\theta_j),
$$

which can be solved for V_s

$$
V_s = \frac{V_{T+1}}{\prod_{v=s}^{T}(1+\theta_v)} + \frac{[\sum_{u=s}^{T}[\frac{(1-\tau_{du})}{(1-\tau_{gu})}]DIV_u - VN_u]\prod_{j=s+1}^{T}(1+\theta_j)}{\prod_{v=s}^{T}(1+\theta_v)}.
$$

Taking the limit as $T \to \infty$ and imposing the transversality condition

$$
\lim_{T\to\infty} V_{T+1} \prod_{u=t}^{T} \frac{1}{(1+\theta_u)} = 0,
$$

which rules out the possibility that the firm may become infinitely large, yields

$$
V_s = \sum_{u=s}^{\infty} \frac{[\frac{(1-\tau_{du})}{(1-\tau_{gu})}]DIV_u - VN_u}{\prod_{v=s}^{u}(1+\theta_v)}, \tag{32}
$$

for $s \geq t$. That is, V_s equals the present value of all future net distributions to shareholders. Note that under the assumption of individual arbitrage, the firm's discount rate $\theta_s = \frac{(1-\tau_{iu})i_s}{(1-\tau_{gu})}$ is increased to reflect the fact that dividend distributions avoid the capital gains tax that arises when the firm retains earnings. Note also that this is a perfect foresight condition in that the firm must predict all future values of the interest rate (in θ_s) and the tax rate variables. The firm's manager chooses L_s and I_s to maximize its market value, as defined in (32), subject to the constraints (17-29).

To simplify the expression for the market value of the firm, substitute from (24), (26) and (28) into expression (27) and simplify to yield

$$
\begin{aligned}
DIV_s &= \zeta[(1-\tau_{bs})(p_sF(K_s, Le_s) - w_sL_s) - (1-\tau_{bs}f_2)i_sB_s \\
&\quad + \tau_{bs}f_1I_s(1+\Phi_s) - f_3\tau_{bs}bI_s + f_4\tau_{bs}\delta bK_s + f_5\tau_{bs}\delta^{\tau}K_s^{\tau} - p_s\delta K_s].
\end{aligned} \tag{33}
$$

Using equation (27) to define dividends paid out as a share of after-tax earnings net of depreciation as follows:

$$
\zeta = \frac{DIV_s}{EARN_s - TE_s - p_s\delta K_s}.
$$

Subtracting one from both sides and dividing by ζ yields

$$\frac{\zeta-1}{\zeta}=\frac{\frac{DIV_s}{EARN_s-TE_s-p_s\delta K_s}-\frac{EARN_s-TE_s-p_s\delta K_s}{EARN_s-TE_s-p_s\delta K_s}}{\frac{DIV_s}{EARN_s-TE_s-p_s\delta K_s}},$$

$$DIV_s(\frac{\zeta-1}{\zeta})=DIV_s-EARN_s+TE_s+p_s\delta K_s. \tag{34}$$

Rearranging the firm's cash flow equation, given in equation (29), to express new share issues in terms of other variables yields

$$VN_s=DIV_s-EARN_s+TE_s+\delta K_s-\delta K_s+I_s(1+\Phi_s)-BN_s.$$

Using equations (34) and (25) to substitute for terms in the equation above yields

$$VN_s=DIV_s(\frac{\zeta-1}{\zeta})-\delta K_s+I_s(1+\Phi_s)-b(I_s-\delta K_s). \tag{35}$$

Substituting equation (35) into equation (32) yields

$$V_s=\sum_{u=s}^{\infty}\frac{DIV_u[\frac{\Omega_u}{\zeta}]-I_u(1+\Phi_u)+\delta K_u+b(I_u-\delta K_u)}{\prod_{v=s}^{u}(1+\theta_v)},$$

where

$$\frac{\Omega_u}{\zeta}=\frac{(1-\tau_{du})}{(1-\tau_{gu})}-\frac{\zeta-1}{\zeta},$$

so that

$$\Omega_u=\frac{[\zeta(1-\tau_{du})+(1-\zeta)(1-\tau_{gu})]}{(1-\tau_{gu})}.$$

Substituting for the value of dividends, as defined in equation (33),and recalling that $B_s=bK_s$, implies that

$$\begin{aligned}V_s \;=\; & \sum_{u=s}^{\infty}\{[\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}][(1-\tau_{bs})\Omega_u(p_sF(K_s,Le_s)-w_sL_s) \\ & -(1-\tau_{bs}f_2)\Omega_u i_s bK_s+f_1\tau_{bs}\Omega_u I_s(1+\Phi_s)-f_3\tau_{bs}\Omega_u bI_s \\ & +f_4\tau_{bs}\Omega_u\delta bK_s+f_5\tau_{bs}\Omega_u\delta^{\tau}K_s^{\tau}-p_s\Omega_u\delta K_s] \\ & -I_u(1+\Phi_u)+\delta K_u+b(I_u-\delta K_u)\}.\end{aligned} \tag{36}$$

The term $f_5\tau_{bs}\Omega_u\delta^{\tau}K_s^{\tau}$ reflects the amount of tax savings from depreciation allowances, which include both those attributable to past investments (previous to time t) and those attributable to future investments made after time t. In order to maximize the value of the firm, the firm's manager chooses the optimal path of future

investments. Accordingly, it is useful to distinguish between the present value of depreciation allowances on old capital, which are irrelevant to this decision, and the present value of depreciation allowances on future investments.

To separate out these two effects from the term $f_5\tau_{bs}\Omega_u\delta^\tau K_s^\tau$ it is necessary to trace out the time path of the tax basis, K^τ. This path differs from the true capital accumulation path due to the modestly accelerated depreciation allowances under current law. For $s \geq t$, the time path of the tax basis, K^τ, follows the equation

$$K_s^\tau = I_{s-1} + (1-\delta^\tau)K_{s-1}^\tau.$$

Substituting for $K_{s-1}^\tau = I_{s-2} + (1-\delta^\tau)K_{s-2}^\tau$ into the previous equation yields

$$K_s^\tau = I_{s-1} + (1-\delta^\tau)I_{s-2} + (1-\delta^\tau)^2 K_{s-2}^\tau.$$

Similarly, substituting for $K_{s-2}^\tau = I_{s-3} + (1-\delta^\tau)K_{s-2}$ yields

$$K_s^\tau = I_{s-1} + (1-\delta^\tau)I_{s-2} + (1-\delta^\tau)^2 I_{s-3} + (1-\delta^\tau)^3 K_{s-3}^\tau.$$

In general (i.e., for $s = t-3$ above, and noting that the summation over v below begins at $v = t = s-(s-t)$ and ends at $v = s-1$),

$$K_s^\tau = (1-\delta^\tau)^{s-t}K_t^\tau + \sum_{j=t}^{s-1}(1-\delta^\tau)^{s-1-j}I_j;$$

that is, the tax basis at time s reflects the basis existing at the beginning of the period in the year of enactment (t) plus all new investment since time t through time $s-1$ (but not s, since the capital stock is measured at the beginning of the period). Thus, the term in equation (36) that reflects the present value of tax savings from all past and future depreciation allowances,

$$\sum_{u=s}^{\infty}[\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}]\tau_{bu}\Omega_u\delta^\tau K_u^\tau \tag{37}$$

can be rewritten as the sum of two terms as follows

$$\begin{aligned} & \{\sum_{u=s}^{\infty}[\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}]\tau_{bu}\Omega_u\delta^\tau\}K_u^\tau \qquad (38) \\ = \; & \{\sum_{u=s}^{\infty}[\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-t}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}]\}K_t^\tau \\ & + \sum_{u=s}^{\infty}\{\tau_{bu}\Omega_u\delta^\tau\sum_{j=t}^{u-1}[(1-\delta^\tau)^{u-1-j}I_j]\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}\}. \end{aligned}$$

The second term in equation (38) represents the tax savings from accelerated depreciation allowances on future investments, and can be expanded (noting that there are no terms in the sum when $s = t$) to yield

$$
\begin{aligned}
& \sum_{u=s}^{\infty}\{\tau_{bu}\Omega_u\delta^\tau \sum_{j=t}^{u-1}[(1-\delta^\tau)^{u-1-j}I_j]\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}\} \\
= \; & \tau_{bs+1}\Omega_{s+1}\delta^\tau \prod_{v=s}^{s+1}\frac{1}{(1+\theta_v)}[I_s] \\
& +\tau_{bs+2}\Omega_{s+2}\delta^\tau \prod_{v=s}^{s+2}\frac{1}{(1+\theta_v)}[I_{s+1}+I_s(1-\delta^\tau)] \\
& +\tau_{bs+3}\Omega_{s+3}\delta^\tau \prod_{v=s}^{s+3}\frac{1}{(1+\theta_v)}[I_{s+2}+I_{s+1}(1-\delta^\tau)+I_s(1-\delta^\tau)^2] \\
& +\tau_{bs+4}\Omega_{s+4}\delta^\tau \prod_{v=s}^{s+4}\frac{1}{(1+\theta_v)}[I_{s+3}+I_{s+2}(1-\delta^\tau)+I_{s+1}(1-\delta^\tau)^2+I_s(1-\delta^\tau)^3]...
\end{aligned}
$$

Collecting terms with I_t yields

$$
\begin{aligned}
& I_s\{\tau_{bs+1}\Omega_{s+1}\delta^\tau \prod_{v=s}^{s+1}\frac{1}{(1+\theta_v)}+(1-\delta^\tau)\tau_{bs+2}\Omega_{s+2}\delta^\tau \prod_{v=s}^{s+2}\frac{1}{(1+\theta_v)} \\
& +(1-\delta^\tau)^2\tau_{bs+3}\Omega_{s+3}\delta^\tau \prod_{v=s}^{s+3}\frac{1}{(1+\theta_v)}+(1-\delta^\tau)^3\tau_{bs+4}\Omega_{s+4}\delta^\tau \prod_{v=s}^{s+4}\frac{1}{(1+\theta_v)}..\} \\
= \; & I_u[\sum_{u=s+1}^{\infty}\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-s-1}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}].
\end{aligned}
$$

Similarly, collecting terms with I_{t+1} yields

$$
\begin{aligned}
& I_{s+1}\{\tau_{bs+2}\Omega_{s+2}\delta^\tau \prod_{v=s}^{s+2}\frac{1}{(1+\theta_v)}+(1-\delta^\tau)\tau_{bs+3}\Omega_{s+3}\delta^\tau \prod_{v=s}^{s+3}\frac{1}{(1+\theta_v)} \\
& +(1-\delta^\tau)^2\tau_{bs+4}\Omega_{s+4}\delta^\tau \prod_{v=s}^{s+4}\frac{1}{(1+\theta_v)}...\} \\
= \; & I_{s+1}[\sum_{u=s+1}^{\infty}\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-s-1}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}].
\end{aligned}
$$

Summing over all such I_s terms yields

$$= \sum_{j=s}^{\infty}\{I_j[\sum_{u=j+1}^{\infty}\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-j-1}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}]\}$$

which can be rewritten as

$$= \sum_{j=s}^{\infty}\{I_j[\sum_{u=j+1}^{\infty}\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-j-1}\prod_{v=j+1}^{u}\frac{1}{(1+\theta_v)}]\prod_{v=s}^{j}\frac{1}{(1+\theta_v)}\}.$$

Repeating from equation (38) above and substituting the new expression for the second term yields

$$\begin{aligned}
&\sum_{u=s}^{\infty}[\prod_{v=s}^{j}\frac{1}{(1+\theta_v)}]\tau_{bu}\Omega_u\delta^\tau K_u^\tau \\
= \;& \{\sum_{u=s}^{\infty}[\prod_{v=s}^{j}\frac{1}{(1+\theta_v)}]\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-t}\}K_t^\tau \\
&+\sum_{j=s}^{\infty}\{I_j[\sum_{u=j+1}^{\infty}\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-j-1}\prod_{v=j+1}^{u}\frac{1}{(1+\theta_v)}]\prod_{v=s}^{j}\frac{1}{(1+\theta_v)}\},
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\sum_{u=s}^{\infty}[\prod_{v=s}^{j}\frac{1}{(1+\theta_v)}]\tau_{bu}\Omega_u\delta^\tau K_u^\tau = X_t + \sum_{j=s}^{\infty}[I_j Z_{j+1}\prod_{v=s}^{j}\frac{1}{(1+\theta_v)}] \qquad (39) \\
= \;& X_t + \sum_{j=s}^{\infty}[I_j Z_{j+1}\prod_{v=s}^{j}\frac{1}{(1+\theta_v)}],
\end{aligned}$$

where

$$Z_j = \sum_{u=j}^{\infty}[\tau_{bu}\Omega_u\delta^\tau(1-\delta^\tau)^{u-j}\prod_{v=j}^{u}\frac{1}{(1+\theta_v)}]$$

and $X_t = Z_t K_t^\tau$. The first term — X_t — is the value of future depreciation deductions on "old" capital existing at the time of reform and the second term is the value of depreciation deductions on all investment made after the enactment of reform.

Substituting from equation (39) into (36) yields

$$V_s = \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}\Gamma_u + f_5 X_t, \qquad (40)$$

where

$$\begin{aligned}\Gamma_u &= (1-\tau_{bu})\Omega_u[p_u F(K_u, Le_u) - w_u L_u] \\ &\quad -K_u[\Omega_u(1-\tau_{bu}f_2)i_u b - \delta(1-b-\Omega_u(1-f_4\tau_{bs}b))] \\ &\quad -I_u[1-b-\Omega_u\tau_{bs}(f_1-f_3 b) - f_5 Z + \Phi_u(1-\tau_{bu}\Omega_u)].\end{aligned}$$

The firm maximizes (40) subject to the constraints

$$K_{s+1} = I_s + (1-\delta)K_s$$

and

$$\lim_{T\to\infty} K_T \geq 0.$$

Given this, the Lagrangian can be defined as

$$\pounds = \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}\Gamma_u + f_5 X_t + q^*_{u+1}[I_u + (1-\delta)K_u - K_{u+1}]$$

$$= \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}\{\Gamma_u + f_5 X_t + q_{u+1}[I_u + (1-\delta)K_u - K_{u+1}]\},$$

where

$$q^*_{u+1} = [\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}]q_{u+1}.$$

The necessary conditions for a maximum are derived as follows. First, the necessary condition with respect to the optimal choice of labor in each period,

$$\frac{\partial \pounds}{\partial Le_s} = (1-\tau_{bs})\Omega_s[p_s\frac{\partial F}{\partial Le_s} - w_s]\sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)} = 0,$$

implies

$$w_s = p_s F_L, \tag{41}$$

that is, in equilibrium the wage rate must equal the value of the marginal product of labor, $p_s F_L$.

Second, the necessary condition with respect to the optimal choice of investment in each period is given by

$$\frac{\partial \pounds}{\partial I_s} = \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}[-1 + b + f_5 Z_{s+1} + \Omega_u\tau_{bs}(f_1 - f_3 b)$$

$$-\Phi_s(1-\tau_{bs}\Omega_s) - I_s(1-\tau_{bs}\Omega_s)\frac{\partial \Phi_s}{\partial I_s} + q_{s+1}] = 0$$

Recalling that $\frac{\partial \Phi_s}{\partial I_s} = \frac{\Phi'_s}{\partial K_s}$, allows us to write this expression as

$$q_{s+1} = 1 - b - f_5 Z_{s+1} - \Omega_u \tau_{bs}(f_1 - f_3 b) + (1-\tau_{bs}\Omega_s)[\Phi_s + \frac{I_s}{K_s}\Phi'_s]. \tag{42}$$

This describes the variable commonly known as Tobin's q — the ratio of the market value of a marginal unit of capital to its replacement cost. It demonstrates that the shadow price of additional capital goods (q_{s+1}) must equal the after-tax marginal cost of capital goods (the right hand side). Since the investment good is the numeraire, the first term in the equation indicates that the shadow price is simply one in the absence of debt and taxes. The second term reflects the financing of a fraction b of the cost of the investment with debt. The third term reflects the reduction in the shadow price of new capital goods due to tax deductions for depreciation. The last term reflects the costs of installing new capital goods with immediate expensing of such adjustment costs. Recalling $[\Phi_s + \frac{I_s}{K_s}\Phi'_s] = p_s\beta[(\frac{I_s}{K_s}) - \mu]$, this equation can be solved to give the optimal investment rate for the firm as

$$\frac{I_s}{K_s} = \frac{(q_{s+1} - 1 + b + f_5 Z_{s+1} + \Omega_u \tau_{bs}(f_1 - f_3 b))}{p_s\beta(1-\tau_{bs}\Omega_s)} + \mu \tag{43}$$

It is desirable to express investment demand as a function the value of the firm, V_s, rather than of q_{s+1}. To do this, note that, as shown by Hayashi (1982) or Keuschnigg (1990), the assumptions of linear homogeneity of the production function and homogeneity of degree zero of the adjustment cost function in investment and capital, imply the following relationship between marginal q and average q(denoted as Q):

$$q_s = \frac{[V_s - X_s]}{K_s}, Q_s = \frac{V_s}{K_s}. \tag{44}$$

Thus, the investment demand function can be written as

$$\frac{I_s}{K_s} = \frac{(\frac{[V_{s+1}-X_{s+1}]}{K_{s+1}} - 1 + b + f_5 Z_{s+1} + \Omega_u \tau_{bs}(f_1 - f_3 b))}{p_s\beta(1-\tau_{bs}\Omega_s)} + \mu \tag{45}$$

Third, noting that the q_{s+1} term in £ above has two terms with K_s, one in the current period and one in the next period, the necessary condition with respect to the optimal capital stock in each period is given by

$$\frac{\partial £}{\partial K_s} = \sum_{u=s}^{\infty}[\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}\{\Omega_s(1-\tau_{bs})p_s\frac{\partial F}{\partial K_s} - [\Omega_s(1-\tau_{bs}f_2)i_s b - \delta(1-b-\Omega_s(1-f_4\tau_{bs}b))]$$

$$-I_s(1-\tau_{bs}\Omega_s)\frac{\partial \Phi_s}{\partial K_s}\} + \prod_{u=t}^{s}\frac{q_{s+1}}{(1+\theta_u)}(1-\delta) - \prod_{u=t}^{s-1}\frac{q_s}{(1+\theta_u)}] = 0,$$

which implies

$$\begin{aligned}\Omega_s(1-\tau_{bs}f_2)i_s b &= \Omega_s(1-\tau_{bs})p_s\frac{\partial F}{\partial K_s}+\delta(1-b-\Omega_s(1-f_4\tau_{bs}b))-I_s(1-\tau_{bs}\Omega_s)\frac{\partial \Phi_s}{\partial K_s}\\ &\quad +q_{s+1}(1-\delta)-q_s(1+\theta_s)).\end{aligned}$$

Recall that $\theta_s=\frac{(1-\tau_{is})i_s}{(1-\tau_{gs})}$, and thus,

$$\begin{aligned}& i_s[\Omega_s(1-\tau_{bs}f_2)b+q_s\frac{(1-\tau_{is})}{(1-\tau_{gs})}]\\ =\ & \Omega_s(1-\tau_{bs})p_s\frac{\partial F}{\partial K_s}+\delta(1-b-\Omega_s(1-f_4\tau_{bs}b))\\ & -I_s(1-\tau_{bs}\Omega_s)\frac{\partial \Phi_s}{\partial K_s}+q_{s+1}(1-\delta)-q_s.\end{aligned}$$

Substituting equations (21) and (22) into the equation above yields

$$\begin{aligned}i_s &= \frac{\Omega_s(1-\tau_{bs})p_s\frac{\partial F}{\partial K_s}+\delta(1-b-\Omega_s(1-f_4\tau_{bs}b))+q_{s+1}(1-\delta)-q_s}{\Omega_s(1-\tau_{bs}f_2)b+q_s\frac{(1-\tau_{is})}{(1-\tau_{gs})}}\\ &\quad +\frac{(1-\tau_{bs}\Omega_s)p_s(\frac{\beta}{2})[(\frac{I_s}{K_s})^2-\mu^2]}{\Omega_s(1-\tau_{bs}f_2)b+q_s\frac{(1-\tau_{is})}{(1-\tau_{gs})}}.\end{aligned} \tag{46}$$

Equation (46) is the Euler equation. It can be written as the following difference equation in q_t,

$$\begin{aligned}q_{s+1}(1-\delta) &= q_s+[\Omega_s(1-\tau_{bs}f_2)b+q_s\frac{(1-\tau_{is})}{(1-\tau_{gs})}]i_s\\ &\quad -\Omega_s(1-\tau_{bs})p_s\frac{\partial F}{\partial K_s}-\delta(1-b-\Omega_s(1-f_4\tau_{bs}b))\\ &\quad -(1-\tau_{bs}\Omega_s)p_s(\frac{\beta}{2})[(\frac{I_s}{K_s})^2-\mu^2]\end{aligned}$$

or

$$\begin{aligned}q_{s+1}(1-\delta) &= q_s[1+\frac{(1-\tau_{is})i_s}{(1-\tau_{gs})}]-\{-\Omega_s(1-\tau_{bs}f_2)i_s b\\ &\quad +\Omega_s(1-\tau_{bs})p_s\frac{\partial F}{\partial K_s}+\delta(1-b-\Omega_s(1-f_4\tau_{bs}b))\\ &\quad +(1-\tau_{bs}\Omega_s)p_s(\frac{\beta}{2})[(\frac{I_s}{K_s})^2-\mu^2]\}\end{aligned}$$

or

$$\begin{aligned} q_{s+1}(1-\delta) &= q_s(1+\theta_s) - \{-\Omega_s(1-\tau_{bs}f_2)i_s b \\ &\quad +\Omega_s(1-\tau_{bs})p_s\frac{\partial F}{\partial K_s} + \delta(1-b-\Omega_s(1-f_4\tau_{bs}b)) \\ &\quad +(1-\tau_{bs}\Omega_s)p_s(\frac{\beta}{2})[(\frac{I_s}{K_s})^2-\mu^2]\} \end{aligned}$$

which can be solved to yield

$$\begin{aligned} q_s &= \sum_{u=s}^{\infty}\{\prod_{v=s}^{u}\frac{1}{(1+\theta_v)}(1-\delta)^{u-t}[\Omega_u(1-\tau_{bu})p_u\frac{\partial F}{\partial K_u} - \Omega_u(1-\tau_{bu}f_2)i_u b \quad (47) \\ &\quad +\delta(1-b-\Omega_u(1-f_4\tau_{bs}b)) + (1-\tau_{bu}\Omega_u)p_u(\frac{\beta}{2})[(\frac{I_u}{K_u})^2-\mu^2]\}. \end{aligned}$$

This equation implies that the shadow price of new capital — q_s — equals the present value of future income, reflecting the productivity of the asset, depreciation allowances, savings in future installation costs and future interest payments. Since $q_s = \frac{[V_s - X_s]}{K_s}$, the value of "marginal" q can be used to solve for average Q as follows

$$Q_s = \frac{V_s}{K_s} = q_s + \frac{X_s}{K_s};$$

that is, average Q equals marginal q plus an adjustment for future depreciation deductions on existing assets. It is also interesting to note that equation (46) can be solved for the user cost of capital developed by Jorgenson (1963) — the minimum return an investment must yield in order to provide the investor with the same rate of return that would be received from lending at the after-tax interest rate, or

$$\begin{aligned} F_K &= \frac{\Omega_u(1-\tau_{bu}f_2)i_u b - \delta(1-b-\Omega_u(1-f_4\tau_{bs}b)) + q_s(1+\theta_s)}{\Omega_s(1-\tau_{bs})p_s} \quad (48) \\ &\quad + \frac{-q_{s+1}(1-\delta) - (1-\tau_{bu}\Omega_u)p_u(\frac{\beta}{2})[(\frac{I_u}{K_u})^2-\mu^2]}{\Omega_s(1-\tau_{bs})p_s}. \end{aligned}$$

Finally, the optimal solution must satisfy

$$\lim_{T\to\infty} K_{T+1} \geq 0; \lim_{T\to\infty} q^*_{T+1} \geq 0 \;\; and \lim_{T\to\infty} K_{T+1}q^*_{T+1} = 0. \quad (49)$$

4.1 Firm Behavior: The Housing Sector

The analysis of the housing sector is similar to that of the non-housing production sector in that the model is also based on the q theory of investment modified to include adjustment costs. In particular, the model explicitly calculates asset values

in the owner-occupied and rental housing sectors, taking account of the existing differences in the tax treatment of the returns to rental and owner-occupied housing under current law. Analogous to the non-housing production sector, housing owner-occupiers and landlords are assumed to act to maximize the value of their housing capital in perfectly competitive markets with no uncertainty. Note that in the derivations provided below, the superscript, $l = O, R$, is used to denote owner-occupied or rental housing.

The production of both types of housing services is assumed to be characterized by the same Cobb-Douglas production function

$$F(K_s^l, Le_s^l) = (K_s^l)^{\alpha_2}(Le_s^l)^{1-\alpha_2}, \qquad 0 < \alpha_2 < 1, \tag{50}$$

where K_s^l denotes inputs of capital and Le_s^l denotes effective labor used in the production of housing services in period s, and α_2 is the capital share parameter in the Cobb-Douglas production function.

Gross investment in period s, I_s^l, equals

$$I_s^l = K_{s+1}^l - (1-\delta^l)K_s^l. \tag{51}$$

where δ^l is the economic rate of depreciation of capital employed in the housing sector. Analogous to the non-housing production sector modelled above, adjustment costs per unit of investment are represented as

$$\Phi_s^l(\frac{I_s^l}{K_s^l}) = \frac{p_s^l(\frac{\beta}{2})(\frac{I_s^l}{K_s^l} - \mu^l)^2}{(\frac{I_s^l}{K_s^l})}, \tag{52}$$

where p_s^l is the price of housing services and β and μ are the adjustment cost parameters. The value of μ^l is set equal to the steady state ratio of gross investment to capital, which is $\mu^l = \delta^l + n + g + ng$.

The framework for determining the value of housing services to homeowners or landlords is similar to that employed for the non-housing firm. However, in the housing sector dividends take the form of the service flows from the housing stock and thus by definition are paid out in full; in addition, new share issues are assumed to be unavailable.[2] Thus, investment in the housing sector is all financed by either debt or equity. Homeowners and landlords are assumed to maintain constant debt to capital ratios, denoted by b^l, so that

$$B_s^l = b^l(K_s^l), \tag{53}$$

where B_s^l is the stock of outstanding debt at time s. New bond issues in period s, BN_s^l, equal the difference between bonds outstanding in the two consecutive periods

[2] Only 4.3 percent of the net capital stock in the rental housing sector is owned by corporations. Thus, it is assumed that the rental housing sector consists entirely of non-corporate businesses so that new share issues are not used to finance rental housing capital.

s and $s+1$, or

$$BN_s^l = B_{s+1}^l - B_s^l.$$

Combining the equation for new bond issues and (53) yields

$$BN_s^l = b^l(K_{s+1}^l - K_s^l) = b^l(I_s^l - \delta^l K_s^l). \tag{54}$$

Note that this formulation implies that existing loans are repaid at the rate of depreciation of the existing capital stock in the housing sector; that is, loans outstanding in period $s+1$ are $B_{s+1}^l = b^l K_{s+1}^l = b^l K_s^l - b^l \delta^l K_s^l + b^l I_s^l$, so that the amount of old debt is $b^l K_s^l$, $b^l \delta^l K_s^l$ is the amount of this existing debt that is retired, and $b^l I_s^l$ is the amount of new debt that is accumulated.

Earnings in the owner-occupied and rental housing sectors in period s, $EARN_s^l$, are defined as the value of housing services less labor cost, real interest payments on total indebtedness, and property tax payments. The earnings equation in each sector is represented as

$$EARN_s^l = p_s^l F(K_s^l, L_s^l) - w_s L_s^l - i_s B_s^l - cK_s^l, \tag{55}$$

where c is the property tax rate imposed on the value of housing capital. Following Goulder (1989), the labor input in the owner-occupied housing sector represents labor used in the production or maintenance of owner-occupied housing services. Earnings in the owner-occupied housing sector are not taxed, but owner-occupiers receive income tax deductions for mortgage interest and property tax payments. The value of these deductions is

$$TE_s^O = \vartheta \tau_{is}[-i_s B_s^O - cK_s^O] \tag{56}$$

where τ_{is} is the tax rate on individual income and ϑ is the fraction of homeowners that itemize.

In the rental housing sector, the total income tax liability of landlords in period s, assuming that adjustment costs and property taxes are fully deductible, is given by

$$\begin{aligned} TE_s^R &= \tau_s^R[p_s^R F(K_s^R, L_s^R) - w_s L_s^R - f_2 i_s B_s^R - mK_s^R \\ &\quad - f_1 I_s^R(1+\Phi_s^R) + f_3 b^R I_s^R - f_4 \delta^R b^R K_s^R - cK_s^R - f_5 \delta^{R\tau} K_s^{R\tau}] \end{aligned} \tag{57}$$

where m is maintenance expenditures as a fraction of the rental capital stock in each period, $\delta^{R\tau}$ is the rate of depreciation allowed for tax purposes, $K_s^{R\tau}$ is the remaining tax basis of the rental housing capital stock, and the tax rate on rental housing income, τ_s^R, is a weighted average of the non-corporate tax rate on landlord profits and the corporate tax rate (to reflect the modest amount of corporate ownership of rental housing capital),

$$\tau_s^R = (\frac{a_l}{a_l + a_c})\tau_{ls}^R + (\frac{a_c}{a_l + a_c})\tau_{cs}^R,$$

where a_l and a_c are the shares of housing services produce in the landlord-owned and corporate-owned sectors respectively.

Again, this formulation accommodates a variety of tax systems. In the initial income tax steady state, $f_1 = f_3 = f_4 = 0$ and $f_2 = f_5 = 1$. Under the Flat Tax, the "R-based" cash flow business tax is modelled by setting $f_1 = 1$ and $f_2 = f_3 = f_4 = f_5 = 0$. Under the "R+F" based version of the cash flow business tax, $f_5 = 0$ and $f_1 = f_2 = f_3 = f_4 = 1$. Also, note that landlords, but not owner-occupiers, are allowed a tax deduction for maintenance and repair expenditures.

Following Goulder and Summers (1989), it is assumed that neither owner-occupied nor rental housing owners accumulate cash; this implies that cash inflows in period s must equal total disbursements in the owner-occupied and rental housing markets, or

$$EARN_s^l + BN_s^l = S_s^l + I_s^l(1 + \Phi_s^l) + TE_s^l \tag{58}$$

for $l = O, R$, where the net housing service flows to owner-occupants and landlords are given by S_s^O and S_s^R, and BN_s^l is new bonds issued in period s. Solving equation (58) for the net service flow of owner-occupied or rental housing, S_s^l, yields

$$S_s^l = EARN_s^l + BN_s^l - I_s^l(1 + \Phi_s^l) - TE_s^l. \tag{59}$$

Note that this formulation implies that a fraction of all marginal investments in the owner-occupied and rental housing sectors must be financed by reductions in the net service flow to owner-occupiers and landlords, since all housing owners are assumed to maintain a constant debt to capital ratio and are not allowed to issue new shares of stock.

Assuming individual level arbitrage implies that the after-tax nominal return to bonds is equal to the net return of either owning and occupying or renting out a house. This condition is represented as

$$(1 - \tau_{is})i_s = \frac{S_s^l + (1 - \tau_{gs}^l)(V_{s+1}^l - V_s^l)}{V_s^l}, \tag{60}$$

where V_s^l is the value of the owner-occupied or rental firm, $(V_{s+1}^l - V_s^l)$ represents the capital gain on owner-occupied or rental housing, and τ_{gs}^l is the effective annual accrual tax rate on capital gains in the owner-occupied or rental housing sector.[3] Solving the difference equation in (60) subject to the following transversality condition

$$\lim_{T\to\infty} V_{T+1}^l \prod_{u=t}^{T} \frac{1}{(1 + \theta_u^l)} = 0, \tag{61}$$

where $\theta_u^l = \frac{(1-\tau_{is})i_s}{(1-\tau_{gs}^l)}$, yields

[3] The tax rate on capital gains takes into account the benefits of deferral of capital gains and the exemption of most gains on owner-occupied housing.

$$V_s^l = \sum_{u=s}^{T} \{ \prod_{v=s}^{u} [\frac{1}{(1+\theta_v^l)}][\frac{1}{(1-\tau_{gs}^l)}] S_u^l \}. \tag{62}$$

That is, the value of owner-occupied or rental housing equals the present value of all future net service flows received by owner-occupiers or landlords. The transversality condition implies the future value of the owner-occupied or rental housing firm is not allowed to become infinitely large.

4.1.1 Further Details on Firm Behavior in the Rental Housing Market

Landlords choose investment (I_s^R) in order to maximize the market value of their rental housing capital (62) subject to constraints (50) through (58).

To find the net service flow in the rental housing sector substitute equations (54), (55), and (57) into equation (59). The net service flow in the rental housing sector is given by

$$\begin{aligned} S_s^R &= p_s^R F(K_s^R, Le_s^R) - w_s Le_s^R - i_s B_s^R - c(p_s^R K_s^R) + b(I_s^R - \delta^R K_s^R) \\ &\quad - I_s^R(1+\Phi_s^R) - \tau^R[p_s^R F(K_s^R, Le_s^R) - w_s Le_s^R - f_1 I_s^R(1+\Phi_s^R) \\ &\quad - f_2 i_s B_s^R + f_3 b I_s^R - f_4 \delta b K_s^R - c(p_s^R K_s^R) - f_5 p_s^R(\delta^{R\tau} K_s^{R\tau})], \end{aligned}$$

rearranging terms yields

$$\begin{aligned} S_s^R &= (1-\tau^R)[p_s^R F(K_s^R, Le_s^R) - w_s Le_s^R - c(p_s^R K_s^R)] \\ &\quad + b(I_s^R - \delta^R K_s^R) - I_s^R(1+\Phi_s^R) + f_1 \tau^R I_s^R(1+\Phi_s^R) \\ &\quad - (1-\tau^R f_2) i_s B_s^R - f_3 \tau^R b I_s^R + f_4 \tau^R \delta b K_s^R + f_5 \tau^R p_s^R(\delta^{R\tau} K_s^{R\tau})]. \end{aligned} \tag{63}$$

Substituting equation (63) into (62), and defining $\Omega_u^R = [1/(1-\tau_{gu}^R)]$ yields

$$\begin{aligned} V_s^R &= \sum_{u=s}^{T} \{ \prod_{v=s}^{u} [\frac{1}{(1+\theta_v^R)}] \Omega_u^R (1-\tau^R)[p_s^R F(K_s^R, Le_s^R) - w_s Le_s^R - c(p_s^R K_s^R)] \\ &\quad + b\Omega_u^R(I_s^R - \delta^R K_s^R) - I_s^R \Omega_u^R(1+\Phi_s^R) + f_1 \tau^R \Omega_u^R I_s^R(1+\Phi_s^R) \\ &\quad - (1-\tau^R f_2)\Omega_u^R i_s B_s^R - f_3 \tau^R \Omega_u^R b I_s^R + f_4 \tau^R \Omega_u^R \delta b K_s^R + f_5 \tau^R p_s^R \Omega_u^R (\delta^{R\tau} K_s^{R\tau})]\}. \end{aligned} \tag{64}$$

The term $f_5 \tau_s^R p_s^R \Omega_u^R \delta^{R\tau} K_s^{R\tau}$ reflects the value of tax savings from depreciation allowances on capital employed in the rental housing sector, including both those attributable to past investments (previous to time t) and those attributable to future investments made after time t. In maximizing the value of rental housing, the landlords choose the optimal path of future investments. Accordingly, it is useful to distinguish between the present value of depreciation allowances on old rental housing capital and the present value of depreciation allowances on future investments in rental housing.

To separate out these two effects from the term $f_5\tau_s^R p_s^R \Omega_u^R \delta^{R\tau} K_s^{R\tau}$, it is necessary to trace out the time path of the tax basis of $K^{R\tau}$. This path differs from the true capital accumulation path due to the modestly accelerated depreciation allowances provided to rental housing under current law. For $s \geq t$, the time path of tax basis $K^{R\tau}$ follows the equation

$$K_s^{R\tau} = I_{s-1}^R + (1-\delta^{R\tau})K_{s-1}^{R\tau}.$$

Substituting $K_{s-1}^{R\tau} = I_{s-2}^R + (1-\delta^{R\tau})K_{s-2}^{R\tau}$ into the previous equation yields

$$K_s^{R\tau} = I_{s-1}^R + (1-\delta^{R\tau})I_{s-2} + (1-\delta^{R\tau})^2 K_{s-2}^{R\tau}.$$

Similarly, substituting $K_{s-2}^{R\tau} = I_{s-3}^R + (1-\delta^{R\tau})K_{s-3}^{R\tau}$ yields

$$K_s^{R\tau} = I_{s-1}^R + (1-\delta^{R\tau})I_{s-2}^R + (1-\delta^{R\tau})^2 I_{s-3}^R + (1-\delta^{R\tau})^3 K_{s-3}^{R\tau}.$$

In general (i.e., for $s = t-3$ above, and noting that the summation over v below begins at $v = t = s-(s-t)$ and ends at $v = s-1$),

$$K_s^{R\tau} = (1-\delta^{R\tau})^{s-t} K_t^{R\tau} + \sum_{j=t}^{s-1} (1-\delta^{R\tau})^{s-1-j} I_j^R;$$

that is, the tax basis at time s reflects the basis existing at the beginning of the period in the year of enactment (t) plus all new investment since time t through time $s-1$ (but not s, since the capital stock is measured at the beginning of the period). Thus, the term $(f_5\tau_s^R p_s^R \Omega_u^R \delta^{R\tau} K_s^{R\tau})$ that reflects the present value of depreciation allowances on past and future investments,

$$\sum_{u=s}^{\infty} [\prod_{v=s}^{u} \frac{1}{(1+\theta_v^R)}] \tau_u^R p_u^R \Omega_u^R \delta^{R\tau} K_u^{R\tau},$$

can be rewritten as the sum of two terms as follows

$$\begin{aligned} & \{\sum_{u=s}^{\infty} [\prod_{v=s}^{u} \frac{1}{(1+\theta_v)}] \tau_u^R p_u^R \Omega_u^R \delta^{R\tau}\} K_u^{R\tau} \qquad (65) \\ = \; & \{\sum_{u=s}^{\infty} [\tau_u^R p_u^R \Omega_u^R \delta^{R\tau} (1-\delta^{\tau})^{u-t} \prod_{v=s}^{u} \frac{1}{(1+\theta_v^R)}]\} K_t^{R\tau} \\ & + \sum_{u=s}^{\infty} \{\sum_{j=t}^{u-1} [\tau_u^R p_u^R \Omega_u^R \delta^{R\tau} (1-\delta^{R\tau})^{u-1-j} I_j^R] \prod_{v=s}^{u} \frac{1}{(1+\theta_v^R)}\}. \end{aligned}$$

The second term in the equation above, the tax savings from depreciation allowances on future investments, can be expanded (noting that there are no terms in the sum when $s = t$) to yield

$$
\begin{aligned}
& \sum_{u=s}^{\infty}\{\tau_u^R p_u^R \Omega_u^R \delta^{R\tau} \sum_{j=t}^{u-1}[(1-\delta^{R\tau})^{u-1-j} I_j^R] \prod_{v=s}^{u} \frac{1}{(1+\theta_v^R)}\} \\
= \; & \tau_{s+1}^R p_{s+1}^R \Omega_{s+1}^R \delta^{R\tau} \prod_{v=s}^{s+1} \frac{1}{(1+\theta_v^R)} [I_s^R] \\
& + \tau_{s+2}^R p_{s+2}^R \Omega_{s+2}^R \delta^{R\tau} \prod_{v=s}^{s+2} \frac{1}{(1+\theta_v^R)} [I_{s+1}^R + I_s^R(1-\delta^{R\tau})] \\
& + \tau_{s+3}^R p_{s+3}^R \Omega_{s+3} \delta^{R\tau} \prod_{v=s}^{s+3} \frac{1}{(1+\theta_v^R)} [I_{s+2}^R + I_{s+1}^R(1-\delta^{R\tau}) + I_s^R(1-\delta^{R\tau})^2] ...
\end{aligned}
$$

Collecting terms with I_t^R yields

$$
\begin{aligned}
& I_s^R\{\tau_{s+1}^R p_{s+1}^R \Omega_{s+1}^R \delta^{R\tau} \prod_{v=s}^{s+1} \frac{1}{(1+\theta_v^R)} \\
& + (1-\delta^{R\tau}) \tau_{s+2}^R p_{s+2}^R \Omega_{s+2}^R \delta^{R\tau} \prod_{v=s}^{s+2} \frac{1}{(1+\theta_v^R)} \\
& + (1-\delta^{R\tau})^2 \tau_{s+3}^R p_{s+3}^R \Omega_{s+3}^R \delta^{R\tau} \prod_{v=s}^{s+3} \frac{1}{(1+\theta_v^R)} \\
& + (1-\delta^{R\tau})^3 \tau_{s+4}^R p_{s+4}^R \Omega_{s+4}^R \delta^{R\tau} \prod_{v=s}^{s+4} \frac{1}{(1+\theta_v^R)} ...\} \\
= \; & I_u^R[\sum_{u=s+1}^{\infty} \tau_u^R p_u^R \Omega_u^R \delta^{R\tau} (1-\delta^{R\tau})^{u-s-1} \prod_{v=s}^{u} \frac{1}{(1+\theta_v^R)}].
\end{aligned}
$$

Similarly, collecting terms with I_{t+1}^R yields

$$
\begin{aligned}
& I_{s+1}^R\{\tau_{s+2}^R p_{s+2}^R \Omega_{s+2}^R \delta^{R\tau} \prod_{v=s}^{s+2} \frac{1}{(1+\theta_v^R)} \\
& + (1-\delta^{R\tau}) \tau_{s+3}^R p_{s+3}^R \Omega_{s+3}^R \delta^{R\tau} \prod_{v=s}^{s+3} \frac{1}{(1+\theta_v^R)} \\
& + (1-\delta^{R\tau})^2 \tau_{s+4}^R p_{s+4}^R \Omega_{s+4}^R \delta^{R\tau} \prod_{v=s}^{s+4} \frac{1}{(1+\theta_v^R)} ...\} \\
= \; & I_{s+1}^R[\sum_{u=s+1}^{\infty} \tau_u^R p_u^R \Omega_u^R \delta^{R\tau} (1-\delta^{R\tau})^{u-s-1} \prod_{v=s}^{u} \frac{1}{(1+\theta_v^R)}].
\end{aligned}
$$

Summing over all such I_s^R terms yields

$$= \sum_{j=s}^{\infty}\{I_j^R[\sum_{u=j+1}^{\infty}\tau_u^R p_u^R \Omega_u^R \delta^{R\tau}(1-\delta^{R\tau})^{u-j-1}\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}]\},$$

which can be written as

$$= \sum_{j=s}^{\infty}\{I_j^R[\sum_{u=j+1}^{\infty}\tau_u^R p_u^R \Omega_u^R \delta^{R\tau}(1-\delta^{R\tau})^{u-j-1}\prod_{v=j+1}^{u}\frac{1}{(1+\theta_v^R)}]\prod_{v=s}^{j}\frac{1}{(1+\theta_v^R)}\}.$$

Repeating from equation (65) above and substituting the new expression for the second term yields

$$\begin{aligned} & \sum_{u=s}^{\infty}[\prod_{v=s}^{j}\frac{1}{(1+\theta_v^R)}]\tau_u^R p_u^R \Omega_u^R \delta^{R\tau} K_u^{R\tau} \qquad (66) \\ = \; & X_t^R + \sum_{j=s}^{\infty}[I_j^R Z_{j+1}^R \prod_{v=s}^{j}\frac{1}{(1+\theta_v^R)}] \\ = \; & X_t^R + \sum_{j=s}^{\infty}[I_j^R Z_{j+1}^R \prod_{v=s}^{j}\frac{1}{(1+\theta_v^R)}] \end{aligned}$$

where

$$Z_j^R = \sum_{u=j}^{\infty}[\tau_u^R p_u^R \Omega_u^R \delta^{R\tau}(1-\delta^{R\tau})^{u-j}\prod_{v=j}^{u}\frac{1}{(1+\theta_v^R)}]$$

and $X_t^R = Z_t^R K_t^{R\tau}$. The first term — X_t^R — is the value of future depreciation deductions on "old" housing capital in the rental sector existing at the time of reform and the second term is the value of depreciation deductions on investments made after the enactment of reform.

Substituting equation (66) into (64) yields

$$V_s = \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}\Gamma_u^R + f_5 X_t^R, \qquad (67)$$

where

$$\begin{aligned} \Gamma_u^R \; = \; & \Omega_u^R(1-\tau_u^R)[p_u^R F(K_u^R, Le_u^R) - w_u Le_u^R] \\ & -K_u^R[\Omega_u^R(1-\tau_u^R)cp_u^R + b^R\Omega_u^R\delta^R - \Omega_u^R(1-\tau_u^R f_2)i_u b^R - f_4\tau^R\Omega_u^R\delta b^R] \\ & -I_u^R[\Omega_u^R - b^R\Omega_u^R - \tau^R\Omega_u^R(f_1 - f_3 b^R) + \Omega_u^R(1-\tau_u^R)\Phi_s^R - f_5 Z_{u+1}^R]. \end{aligned}$$

Thus, landlords maximize (67) subject to the constraints

$$K_{s+1}^R = I_s^R + (1-\delta^R)K_s^R$$

and

$$\lim_{T\to\infty} K_T^R \geq 0.$$

The Lagrangian for the rental housing market is

$$\pounds^R = \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}\Gamma_u^R + f_5 X_t^R + q_{u+1}^{R*}[I_u^R + (1-\delta^R)K_u^R - K_{u+1}^R]$$

$$= \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}\{\Gamma_u^R + f_5 X_t^R + q_{u+1}^{R}[I_u^R + (1-\delta^R)K_u^R - K_{u+1}^R]\},$$

where

$$q_{u+1}^{R*} = [\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}]q_{u+1}^R.$$

The necessary conditions for a maximum are derived as follows.

First, the necessary condition with respect to the optimal choice of investment in each period is

$$\begin{aligned}\frac{\partial \pounds}{\partial I_s^R} &= \sum_{u=s}^{\infty}\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}\{b^R\Omega_u^R - \Omega_u^R + \tau^R\Omega_u^R(f_1 - f_3 b^R) \\ &\quad + f_5 Z_{s+1}^R - (1-\tau_s^R)\Omega_s^R\Phi_s^R - (1-\tau_s^R)\Omega_s^R\frac{\partial\Phi_s^R}{\partial I_s^R} + q_{s+1}^R\}.\end{aligned}$$

Recall that $\frac{\partial\Phi_s^R}{\partial I_s^R} = \frac{\Phi_s'^R}{K_s^R}$, this implies

$$q_{s+1}^R = \Omega_u^R - b^R\Omega_u^R - \tau^R\Omega_u^R(f_1 - f_3 b^R) - f_5 Z_{s+1}^R + (1-\tau_s^R)\Omega_s^R[\Phi_s^R + \frac{\Phi_s'^R}{K_s^R}]. \quad (68)$$

Again, this describes the variable commonly known as Tobin's q. It demonstrates that the shadow price of additional rental housing capital (q_{s+1}^R) must equal the after-tax marginal cost of rental housing capital (the right hand side). The first term indicates that the shadow price is simply one in the absence of debt and taxes, since the investment good is the numeraire. The second term reflects the financing of a fraction b of the cost of the investment with debt. The third term reflects the reduction in the shadow price of new capital goods due to tax deductions for depreciation. The last term reflects the costs of installing new capital goods with immediate expensing

of such adjustment costs. Recalling that $[\Phi_s + \frac{I_s}{K_s}\Phi'_s] = p_s\beta[(\frac{I_s^R}{K_s^R}) - \mu]$, this equation can be solved to give the optimal investment rate for the firm as

$$\frac{I_s^R}{K_s^R} = [\frac{q_{s+1}^R - \Omega_u^R + b^R\Omega_u^R + \tau^R\Omega_u^R(f_1 - f_3b^R) + f_5Z_{s+1}^R}{(1-\tau_s^R)\Omega_s^R p_s\beta}] + \mu. \tag{69}$$

In order to express investment demand as a function of the value of the firm, V_s^R, rather than q_{s+1}^R, the relationship between marginal q and average q (denoted as Q), as shown by Hayashi (1982), is defined as

$$q_s^R = \frac{[V_s^R - X_s^R]}{K_s^R},\ Q_s^R = \frac{V_s^R}{K_s^R}.$$

Thus, the investment demand function can be written as

$$\frac{I_s^R}{K_s^R} = [\frac{\frac{[V_{s+1}^R - X_{s+1}^R]}{K_{s+1}^R} - \Omega_u^R + b^R\Omega_u^R + \tau^R\Omega_u^R(f_1 - f_3b^R) + f_5Z_{s+1}^R}{(1-\tau_s^R)\Omega_s^R p_s\beta}] + \mu.$$

Noting that the q_{s+1}^Rterm in the Lagrangian above has two terms with K_s^R, one in the current period and one in the next period, the necessary condition with respect to the optimal capital stock in each period,

$$\frac{\partial \pounds}{\partial K_s} = \sum_{u=s}^{\infty}[\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}\{\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} - [cp_u^R + b^R\Omega_u^R\delta^R - \Omega_u^R(1-\tau_u^R)i_ub^R]$$

$$-I_s^R\Omega_s^R(1-\tau_s^R)\frac{\partial \Phi_s^R}{\partial K_s^R}\} + \prod_{u=t}^{s}\frac{q_{s+1}^R}{(1+\theta_u^R)}(1-\delta^R) - \prod_{u=t}^{s-1}\frac{q_s^R}{(1+\theta_u^R)}] = 0,$$

which implies

$$\begin{aligned}\Omega_u^R(1-\tau_u^R)i_ub^R &= -\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} + cp_u^R + b^R\Omega_u^R\delta^R \\ &\quad +I_s^R\Omega_s^R(1-\tau_s^R)\frac{\partial \Phi_s^R}{\partial K_s^R} - q_{s+1}^R(1-\delta^R) + q_s^R(1+\theta_s^R)).\end{aligned}$$

Recall that $\theta_s^R = \frac{r_s}{(1-\tau_{gs}^R)}$, this implies

$$\begin{aligned}i_s[\Omega_s^R(1-\tau_s^R)b^R + q_s^R\frac{(1-\tau_{is}^R)}{(1-\tau_{gs}^R)}] &= -\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} + cp_u^R + b^R\Omega_u^R\delta^R \\ &\quad +I_s^R\Omega_s^R(1-\tau_s^R)\frac{\partial \Phi_s^R}{\partial K_s^R} - q_{s+1}^R(1-\delta^R) + q_s^R.\end{aligned}$$

Noting that

$$\frac{\partial \Phi_s^R}{\partial K_s^R} = -(\frac{I_s^R}{(K_s^R)^2})\Phi_s'^R$$

and

$$\Phi_s'^R(\frac{I_s^R}{K_s^R}) = \frac{\partial \Phi_s^R}{\partial(\frac{I_s^R}{K_s^R})} = p_s^R(\frac{\beta}{2})[1-(\frac{\mu}{\frac{I_s^R}{K_s^R}})^2],$$

this can be written as

$$i_s = \frac{-\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} + cp_u^R + b^R\Omega_u^R\delta^R - q_{s+1}^R(1-\delta^R) + q_s^R}{\Omega_s^R(1-\tau_s^R)b^R + q_s^R\frac{(1-\tau_{is}^R)}{(1-\tau_{gs}^R)}} \tag{70}$$

$$+\frac{\Omega_s^R(1-\tau_s^R)p_s^R(\frac{\beta}{2})[(\frac{I_s^R}{K_s^R})^2-\mu^2]}{\Omega_s^R(1-\tau_s^R)b^R + q_s^R\frac{(1-\tau_{is}^R)}{(1-\tau_{gs}^R)}} \tag{71}$$

Equation (70) is the Euler equation for the rental housing sector. It can be written as the following difference equation in q_t,

$$\begin{aligned} q_{s+1}^R(1-\delta^R) &= q_s^R - [\Omega_s^R(1-\tau_s^R)b^R + q_s^R\frac{(1-\tau_{is}^R)}{(1-\tau_{gs}^R)}]i_s \\ &\quad -\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} + cp_u^R + b^R\Omega_u^R\delta^R \\ &\quad +\Omega_s^R(1-\tau_s^R)p_s^R(\frac{\beta}{2})[(\frac{I_s^R}{K_s^R})^2-\mu^2], \end{aligned}$$

or

$$\begin{aligned} q_{s+1}^R(1-\delta^R) &= q_s^R[1-\frac{(1-\tau_{is}^R)}{(1-\tau_{gs}^R)}i_s] - \Omega_s^R(1-\tau_s^R)b^R i_s \\ &\quad -\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} + cp_u^R + b^R\Omega_u^R\delta^R \\ &\quad +\Omega_s^R(1-\tau_s^R)p_s^R(\frac{\beta}{2})[(\frac{I_s^R}{K_s^R})^2-\mu^2], \end{aligned}$$

or

$$\begin{aligned} q_{s+1}^R(1-\delta^R) &= q_s^R[1-\theta_s^R] \\ &\quad -\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} + cp_u^R + b^R\Omega_u^R\delta^R \\ &\quad +\Omega_s^R(1-\tau_s^R)p_s^R(\frac{\beta}{2})[(\frac{I_s^R}{K_s^R})^2-\mu^2], \end{aligned}$$

which can be solved to yield

$$q_s^R = \sum_{u=s}^{\infty}\{\prod_{v=s}^{u}\frac{1}{(1+\theta_v^R)}(1-\delta^R)^{u-t}[-\Omega_s^R(1-\tau_s^R)p_s^R\frac{\partial F}{\partial K_s^R} - \Omega_s^R(1-\tau_s^R)b^R i_s$$
$$+cp_u^R + b^R\Omega_u^R\delta^R + \Omega_s^R(1-\tau_s^R)p_s^R(\frac{\beta}{2})[(\frac{I_s^R}{K_s^R})^2 - \mu^2]\}.$$

This equation implies that the shadow price of new rental housing capital — q_t^R — equals the present value of future income, reflecting the productivity of the asset, depreciation allowances, savings in future installation costs, property tax cost, and future interest payments. Since $q_s^R = \frac{[V_s^R - X_s^R]}{K_s^R}$, this value of "marginal" q can be used to solve for average Q as follows

$$Q_s^R = \frac{V_s^R}{K_s^R} = q_s^R + \frac{X_s^R}{K_s^R};$$

that is, average Q equals marginal q plus an adjustment for future depreciation deductions on existing assets. It is also interesting to note that equation (46) can be solved for the user cost of capital developed by Jorgenson (1963) — the minimum return the investment must yield in order to provide the investor with the same rate of return that would be received from lending at the after-tax interest rate, or

$$F_K = \frac{\Omega_u(1-\tau_{bu})i_u b - \delta(1-b-\Omega_u) + q_s(1+\theta_s) - q_{s+1}(1-\delta)}{\Omega_s(1-\tau_{bs})p_s} - \frac{(1-\tau_{bu}\Omega_u)p_u(\frac{\beta}{2})[(\frac{I_u}{K_u})^2 - \mu^2]}{\Omega_s(1-\tau_{bs})p_s}. \tag{72}$$

Finally, the optimal solution must satisfy

$$\lim_{T\to\infty} K_{T+1}^R \geq 0; \lim_{T\to\infty} q_{T+1}^{*R} \geq 0 \;\; and \lim_{T\to\infty} K_{T+1}^R q_{T+1}^{*R} = 0. \tag{73}$$

4.1.2 Further Details on the Behavior of Owner-Occupiers

As noted above, owner-occupiers are treated as firm owners, analogous to landlords in the rental housing market, except that they "rent" their homes to themselves. To find the expression for the net service flow in the owner-occupied sector, substitute equations (53) and (56) into (59) to yield

$$S_s^o = p_s^o F(K_s^o, Le_s^o) - w_s Le_s^o - (1-\tau_{is})[b^o(K_s^o) + c(p_s^o K_s^o)] + b(I_s^o - \delta^o K_s^o) - I_s^o(1+\Phi_s^o). \tag{74}$$

Substituting equation (74) into (62) and defining $\Omega_u^o = [(1-\tau_u^o)/(1-\tau_{gu}^o)]$ yields

$$V_s = \sum_{u=s}^{\infty} \prod_{v=s}^{u} \frac{1}{(1+\theta_v^o)} \Gamma_u^o, \tag{75}$$

where

$$\begin{aligned} \Gamma_u^o &= \Omega_u^o[p_s^o F(K_s^o, Le_s^o) - w_s Le_s^o] - K_s^o[\Omega_u^o(1-\tau_{is}) i_s b^o + \Omega_u^o(1-\tau_{is}) c b^o p_s^o - \Omega_u^o b \delta^o] \\ &\quad + I_s^o[\Omega_u^o b^o - \Omega_u^o(1+\Phi_s^o)] \end{aligned}$$

Owner-occupiers maximize (75) subject to the constraints

$$K_{s+1}^o = I_s^o + (1-\delta^o) K_s^o$$

and

$$\lim_{T\to\infty} K_T^o \geq 0.$$

The Lagrangian for the owner-occupied housing market is

$$\begin{aligned} \pounds^o &= \sum_{u=s}^{\infty} \prod_{v=s}^{u} \frac{1}{(1+\theta_v^o)} \Gamma_u^o + q_{u+1}^{o*}[I_u^o + (1-\delta^o) K_u^o - K_{u+1}^o] \\ &= \sum_{u=s}^{\infty} \prod_{v=s}^{u} \frac{1}{(1+\theta_v^o)} \{\Gamma_u^o + q_{u+1}^o[I_u^o + (1-\delta^o) K_u^o - K_{u+1}^o]\}, \end{aligned}$$

where

$$q_{u+1}^{o*} = [\prod_{v=s}^{u} \frac{1}{(1+\theta_v^o)}] q_{u+1}^o.$$

The necessary conditions for a maximum are derived as follows.

First, the necessary condition with respect to the optimal choice of investment in each period is

$$\frac{\partial \pounds}{\partial I_s^o} = \sum_{u=s}^{\infty} \prod_{v=s}^{u} \frac{1}{(1+\theta_v^o)} \{\Omega_u^o b^o - \Omega_u^o(1+\Phi_s^o) - I_s^o \Omega_u^o \frac{\partial \Phi_s^o}{\partial I_s^o} + q_{u+1}^o\} = 0.$$

where $\frac{\partial \Phi_s^o}{\partial I_s^o} = \frac{\Phi_s^{'o}}{K_s^o}$, this implies

$$q_{s+1}^o = \Omega_u^o - b^o \Omega_u^o + \Omega_s^o[\Phi_s^o + (\frac{I_s^o}{K_s^o}) \Phi_s^{'o}]. \tag{76}$$

Again, this describes the variable commonly known as Tobin's q. It demonstrates that the shadow price of additional owner-occupied housing capital (q_{s+1}^o) must equal the after-tax marginal cost of owner-occupied housing capital (the right hand side). Since the investment good is the numeraire, the shadow price is simply one in the

absence of debt and taxes. The second term reflects the financing of a fraction b of the cost of the investment with debt. The third term reflects the reduction in the shadow price of new capital goods due to tax deductions for depreciation. The last term reflects the costs of installing new capital goods with immediate expensing of such adjustment costs. Recalling that $[\Phi_s^o + \frac{I_s}{K_s}\Phi_s^{o\prime}] = p_s^o\beta[(\frac{I_s^o}{K_s^o}) - \mu]$, this equation can be solved to give the optimal investment rate for the firm according to

$$\frac{I_s^o}{K_s^o} = [\frac{q_{s+1}^o - \Omega_u^o + b^o\Omega_u^o}{\Omega_s^o p_s^o \beta}] + \mu. \tag{77}$$

In order to express investment demand as a function of the value of the firm, V_s^o, rather than q_{s+1}^o, the relationship between marginal q and average q (denoted as Q) as shown by Hayashi (1982) or Keuschnigg (1990) and described previously in the description of firm behavior is used. This implies

$$q_s^o = \frac{[V_s^o - X_s^o]}{K_s^o},\ Q_s^o = \frac{V_s^o}{K_s^o}.$$

Thus, the investment demand function can be written as

$$\frac{I_s^o}{K_s^o} = [\frac{\frac{[V_{s+1}^o - X_{s+1}^o]}{K_{s+1}^o} - \Omega_u^o + b^o\Omega_u^o}{\Omega_s^o p_s^o \beta}] + \mu.$$

Noting that the q_{s+1}^oterm in the Lagrangian above has two terms with K_s^o, one in the current period and one in the next period, the necessary condition with respect to the optimal capital stock in each period,

$$\frac{\partial \pounds}{\partial K_s^o} = \sum_{u=s}^{\infty}[\prod_{v=s}^{u}\frac{1}{(1+\theta_v^o)}\{\Omega_s^o p_s^o \frac{\partial F}{\partial K_s^o} - [\Omega_u^o(1-\tau_{iu})i_u b^o + \Omega_u^o(1-\tau_{iu})cb^o p_s^o - \Omega_u^o b^o \delta^o]$$

$$-I_s^o\Omega_s^o\frac{\partial \Phi_s^o}{\partial K_s^o}\} + \prod_{u=t}^{s}\frac{q_{s+1}^o}{(1+\theta_u^o)}(1-\delta^o) - \prod_{u=t}^{s-1}\frac{q_s^o}{(1+\theta_u^o)}] = 0,$$

which implies

$$\begin{aligned}\Omega_u^o(1-\tau_u^o)i_u b^o &= \Omega_s^o p_s^o\frac{\partial F}{\partial K_s^o} - \Omega_u^o(1-\tau_{iu})cb^o p_s^o + \Omega_u^o b^o\delta^o + I_s^o\Omega_s^o\frac{\partial \Phi_s^o}{\partial K_s^o} \\ &\quad -q_{s+1}^o(1-\delta^o) + q_s^o(1+\theta_s^o).\end{aligned}$$

Recall that $\theta_s^o = \frac{r_s}{(1-\tau_{gs}^o)}$,and thus,

$$\begin{aligned}i_s[\Omega_s^o(1-\tau_s^o)b^o + q_s^o\frac{(1-\tau_{is})}{(1-\tau_{gs}^o)}] &= \Omega_s^o p_s^o\frac{\partial F}{\partial K_s^o} - \Omega_u^o(1-\tau_{iu})cb^o p_s^o \\ &\quad +\Omega_u^o b^o\delta^o + I_s^o\Omega_s^o\frac{\partial \Phi_s^o}{\partial K_s^o} - q_{s+1}^o(1-\delta^o) + q_s^o.\end{aligned}$$

Since

$$\frac{\partial \Phi_s^o}{\partial K_s^o} = -\frac{I_s^o}{((K_s^o)^2})\Phi_s'^o$$

and

$$\Phi_s'^o(\frac{I_s^o}{K_s^o}) = \frac{\partial \Phi_s^o}{\partial(\frac{I_s^o}{K_s^o})} = p_s^o(\frac{\beta}{2})[1-(\frac{\mu}{\frac{I_s^o}{K_s^o}})^2],$$

this can be written as

$$i_s = \frac{\Omega_s^o p_s^o \frac{\partial F}{\partial K_s^o} - \Omega_u^o(1-\tau_{iu})cb^o p_s^o + \Omega_u^o b^o \delta^o - q_{s+1}^o(1-\delta^o) + q_s^o - \Omega_s^o p_s^o(\frac{\beta}{2})[(\frac{I_s^o}{K_s^o})^2 - \mu]}{\Omega_s^o(1-\tau_s^o)b^o + q_s^o\frac{(1-\tau_{is})}{(1-\tau_{gs}^o)}}. \tag{78}$$

Equation (78) is the Euler equation for owner-occupied housing. It can be written as the following difference equation in q_t,

$$\begin{aligned} q_{s+1}^o(1-\delta^o) &= q_s^o - [\Omega_s^o(1-\tau_s^o)b^o + q_s^o\frac{(1-\tau_{is})}{(1-\tau_{gs}^o)}]i_s \\ &\quad -\Omega_s^o p_s^o\frac{\partial F}{\partial K_s^o} + \Omega_u^o(1-\tau_{iu})cb^o p_u^o + \Omega_u^o b^o\delta^o \\ &\quad +\Omega_s^o p_s^o(\frac{\beta}{2})[(\frac{I_s^o}{K_s^o})^2 - \mu^2], \end{aligned}$$

or

$$\begin{aligned} q_{s+1}^o(1-\delta^o) &= q_s^o[1-\frac{(1-\tau_{is})}{(1-\tau_{gs}^o)}i_s] - \Omega_s^o(1-\tau_s^o)b^o i_s \\ &\quad -\Omega_s^o p_s^o\frac{\partial F}{\partial K_s^o} + \Omega_u^o(1-\tau_{iu})cb^o p_u^o + \Omega_u^o b^o\delta^o \\ &\quad +\Omega_s^o p_s^o(\frac{\beta}{2})[(\frac{I_s^o}{K_s^o})^2 - \mu^2], \end{aligned}$$

or

$$\begin{aligned} q_{s+1}^o(1-\delta^o) &= q_s^o[1-\theta_s^o] - \Omega_s^o(1-\tau_s^o)b^o i_s \\ &\quad -\Omega_s^o p_s^o\frac{\partial F}{\partial K_s^o} + \Omega_u^o(1-\tau_{iu})cb^o p_u^o + \Omega_u^o b^o\delta^o \\ &\quad +\Omega_s^o p_s^o(\frac{\beta}{2})[(\frac{I_s^o}{K_s^o})^2 - \mu^2], \end{aligned}$$

which can be solved to yield

$$
\begin{aligned}
q_s^o &= \sum_{u=s}^{\infty}\{\prod_{v=s}^{u}\frac{1}{(1+\theta_v^o)}(1-\delta^o)^{u-t}[\Omega_s^o p_s^o\frac{\partial F}{\partial K_s^o}-\Omega_s^o(1-\tau_s^o)b^o i_s \\
&\quad +\Omega_u^o(1-\tau_{iu})cb^o p_u^o+\Omega_u^o b^o\delta^o+\Omega_s^o p_s^o(\frac{\beta}{2})[(\frac{I_s^o}{K_s^o})^2-\mu^2]\}.
\end{aligned}
$$

This equation implies that the shadow price of new owner-occupied housing capital — q_t^o — equals the present value of future income, reflecting the productivity of the asset, depreciation allowances, savings in future installation costs, property tax costs, and future interest payments. Since $q_s^o = \frac{[V_s^o - X_s^o]}{K_s^o}$, this value of "marginal" q can be used to solve for average Q as

$$
Q_s^o = \frac{V_s^o}{K_s^o} = q_s^o + \frac{X_s^o}{K_s^o};
$$

that is, average Q equals marginal q plus an adjustment for future depreciation deductions on existing assets. It is also interesting to note that equation (78) can be solved for the user cost of capital developed by Jorgenson (1963) — the minimum return the investment must yield in order to provide the investor with the same rate of return that would be received from lending at the after-tax interest rate, or

$$
\begin{aligned}
F_K &= \frac{\Omega_u(1-\tau_{bu})i_u b-\delta(1-b-\Omega_u)+q_s(1+\theta_s)-q_{s+1}(1-\delta)}{\Omega_s(1-\tau_{bs})p_s} \\
&\quad -\frac{(1-\tau_{bu}\Omega_u)p_u(\frac{\beta}{2})[(\frac{I_u}{K_u})^2-\mu^2]}{\Omega_s(1-\tau_{bs})p_s},
\end{aligned} \tag{79}
$$

with the optimal solution satisfying

$$
\lim_{T\to\infty} K_{T+1}^o \geq 0; \lim_{T\to\infty} q_{T+1}^{*o} \geq 0 \;\; and \lim_{T\to\infty} K_{T+1}^o q_{T+1}^{*o} = 0. \tag{80}
$$

5 Market Equilibria

Finally, prices must adjust so that equilibrium is obtained in all periods in all markets. This requires that aggregate demand (including investment demands, inclusive of adjustment costs, and government demands) equal aggregate supply for all three goods (the composite consumption good, owner-occupied housing, and rental housing), that aggregate demand equal aggregate supply in the markets for capital and labor, and that the government budget (including both national and state governments) be balanced in each period (with the option of increasing government debt for a finite period). Asset market equilibrium requires that the total equity holdings of

individuals equals the total firm value for all three types of firms and that the total debt holdings of individuals equals the amount of debt issued by all three types of firms. Implicit in the calculation of these equilibria is that the expectations regarding all future prices by all individuals and firms must be satisfied.